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Asymptotic analysis of the linearized Navier-Stokes equation on an exterior circular domain : explicit solution and the zero viscosity limit.

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Abstract

In this paper we study and derive explicit formulas for the linearized Navier-Stokes equations on an exterior circular domain in space dimension two.

Through an explicit construction, the solution is decomposed in an inviscid solution, a boundary layer solution and a corrector. Bounds on these solutions are given, in the appropriate Sobolev spaces, in terms of the norms of the initial and boundary data. The correction term is shown to be of the same order of magnitude of the square root of the viscosity.

1 Introduction

In this paper we shall investigate the time dependent incompressible Stokes equations on an exterior circular domain in space dimension two, i.e.:

$$\partial_t u_\phi + r^{-1}\partial_\phi p = \nu\left(\Delta u_\phi - r^{-2}u_\phi + 2r^{-2}\partial_\phi u_r\right) , \tag{1.1}$$

$$\partial_t u_r + \partial_r p = \nu\left(\Delta u_r - r^{-2}u_r - 2r^{-2}\partial_\phi u_\phi\right) , \tag{1.2}$$

$$\nabla \cdot \boldsymbol{u} = 0 , \tag{1.3}$$

$$\gamma_R \boldsymbol{u} = \boldsymbol{g} , \tag{1.4}$$

$$\boldsymbol{u}(r,\phi,t=0) = \boldsymbol{u}_0(r,\phi) , \tag{1.5}$$

where $\boldsymbol{u} = (u_r(r,\phi,t), u_\phi(r,\phi,t))$ is the velocity field which depends on the radial variable r with $r \geq R$, on the angular variable ϕ with $0 \leq \phi \leq 2\pi$ and on the time variable $t \geq 0$. In the Eqs. (1.1)-(1.5) $\Delta = r^{-1}\partial_r(r\partial_r) + r^{-2}\partial_{\phi\phi}$, $\nabla \cdot \boldsymbol{u} = r^{-1}\partial_r(ru_r) + r^{-1}\partial_\phi u_\phi$, $p = p(r,\phi,t)$ is the pressure, $\nu = \varepsilon^2$ is the viscosity coefficient and γ_R is the trace operator defined by $\gamma_R f(r,\phi,t) = f(R,\phi,t)$.

Equations (1.1) and (1.2) are the conservation of momentum for a viscous fluid obtained by linearizing the Navier-Stokes equation, Eq. (1.3) is the incompressibility condition and Eqs. (1.4) and (1.5) are the boundary and the initial conditions respectively.

We are interested in the behavior of the solution in the limit of small viscosity. In fact the problem of the convergence of the Navier-Stokes equations to the Euler equation in presence

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of boundaries is a relevant problem in fluid dynamics and Stokes equations can be considered a good simplified mathematical model that can be helpful in the understanding of the more complicated nonlinear case.

When the Reynolds number is large the fluid shows two different regimes: far away from the boundary the viscous forces are negligible with respect to the inertial forces and the behavior of the fluid is believed to be well described by the Euler equations. They are obtained neglecting the viscosity term. As a consequence of the change in the order of the equations, only the no-flux boundary condition can be imposed. In the vicinity of the boundary, on the other hand, the viscous forces are not negligible and a boundary layer whose thickness is proportional to the square root of the viscosity appears. The no-slip condition causes a rapid variation of the tangential component of the velocity to adjust the flow to the value given by the inviscid outer theory. In the boundary layer the fluid is ruled by the Prandtl equations which are obtained rescaling the normal variable with the square root of the viscosity ($\varepsilon = \sqrt{\nu}$) and imposing that the derivative with respect to the rescaled variable of the tangential velocity is $O(1)$. Therefore, both Euler and Prandtl equations can be obtained from Navier-Stokes through formal asymptotic expansions.

In [2] it was proved that, for analytic solutions of the Navier-Stokes equations on the half space and for a short time, these approximations are indeed correct. The solution of Navier-Stokes equations was constructed as the sum of the Euler solution, the Prandtl solution and a correction which was proved to be vanishingly small with the square root of the viscosity. The initial data were restricted to be analytic.

In this paper we shall be concerned with the incompressible Stokes equations on an exterior circular domain. Following the technique used by Ukai (see [3]) we will give an explicit formula for the solution and then we will use it to perform the asymptotic analysis in the limit of vanishing viscosity.

We will show that the solution of the Stokes equations can be written in the form :

$$\boldsymbol{u}^S = \boldsymbol{u}^E + \boldsymbol{u}^P + \varepsilon \boldsymbol{w} \tag{1.6}$$

where $\boldsymbol{u}^E$ is the solution of the linearized Euler equations, $\boldsymbol{u}^P$ is the Prandtl solution exponentially decaying outside the boundary layer, and $\varepsilon \boldsymbol{w}$ is the correction term. The explicit expression of all the terms of this expansion is given. This result is stated in the Theorem 4.1 below, which is the main result of this paper.

The paper is organized as follows: in Section 2 we shall introduce the spaces of functions we shall be using through the rest of this paper. In Section 3 we shall give the explicit formulas for the solution of the Stokes problem in the exterior circular domain. In Section 4 we shall perform the asymptotic analysis of the Stokes equations and give the appropriate estimates in terms of the initial and boundary data. In particular the norm of the term $\boldsymbol{w}$ in the expansion (1.6) is shown to be $O(1)$.

In this paper all the estimates are given in the usual Sobolev spaces. Moreover the asymptotic expansion (1.6) is shown to be valid for an arbitrarily large time T.

2 Function spaces

In this section we define some function spaces we shall be using through the rest of this paper. All functions depending on the angular variable ϕ will be periodic in this variable. We recall that the L^2 norm expressed in polar coordinates (r, ϕ) is weighted with r. We define the space L_r^2 as the space of those functions $f(r, \phi)$ such that

$$\|f\|_r = \left\{ \int_0^{2\pi} \int_R^{\infty} d\phi \; dr \; r \; |f(r, \phi)|^2 \right\}^{1/2} < \infty.$$

We first introduce the ambient spaces for the inviscid equation.

Definition 2.1 *H'^l is the set of all functions $f(\phi)$ such that*

- $\partial_\phi^j f \in L^2([0, 2\pi])$ *with* $j \leq l$.

We shall denote the usual norm in H'^l with $|f|_l$.

Definition 2.2 *$H_T'^l$ is the set of all functions $f(\phi, t)$ such that*

- $\partial_t^j f(\phi, t) \in L^\infty([0, T], H'^{l-j})$ *with* $j \leq l$.

The norm of $f \in H_T'^l$ is given by:

$$|f|_{l,T} = \sum_{j_1 + j_2 \leq l} \sup_{0 \leq t \leq T} \|\partial_t^{j_1} \partial_\phi^{j_2} f(\cdot, t)\|_{L^2([0,2\pi])}. \tag{2.1}$$

Definition 2.3 *H^l is the set of all functions $f(r, \phi)$ defined on $\Omega = [R, \infty) \times [0, 2\pi]$ such that*

- $r^{-m} \partial_\phi^m \partial_r^j f \in L_r^2$ *with* $m + j \leq l$.

The norm of f is given by:

$$|f|_l = \sum_{m+j \leq l} \|r^{-m} \partial_\phi^m \partial_r^j f(\cdot, \cdot)\|_r. \tag{2.2}$$

Definition 2.4 *H_T^l is the set of all functions $f(r, \phi, t)$ such that*

- $\partial_t^j f(r, \phi, t) \in L^\infty([0, T], H^{l-j})$ *with* $j \leq l$.

The norm of $f \in H_T^l$ is given by:

$$|f|_{l,T} = \sum_{j_1 + j_2 + j_3 \leq l} \sup_{0 \leq t \leq T} \|r^{-j_1} \partial_\phi^{j_1} \partial_t^{j_2} \partial_r^{j_3} f(\cdot, \cdot, t)\|_r. \tag{2.3}$$

We now introduce the ambient spaces for Prandtl equations. All the functions belonging to these spaces depend on the normal scaled variable $Y = \frac{r-R}{\varepsilon}$ and are exponentially decaying with respect to Y. We require differentiability with respect to this variable only up to the second order.

Definition 2.5 *$K^{l,\mu}$ with $\mu > 0$ is the set of all functions $f(Y, \phi)$ such that*

- $\partial_\phi^{j_1} \partial_Y^{j_2} f(Y, \phi) \in L^2([0, 2\pi])$, *with* $j_2 \leq 2$ *and* $j_1 + j_2 \leq l$, *with* $j_1 \leq l - 2$ *if* $j_2 > 0$.

- $\sup_{Y\geq 0} e^{\mu Y}\|\partial_\phi^{j_1}\partial_Y^{j_2} f(Y,\cdot)\|_{L^2} < \infty$ *with* $j_2 \leq 2$ *and* $j_1 + j_2 \leq l$, *with* $j_1 \leq l-2$ *if* $j_2 > 0$.

The norm is given by:

$$\begin{aligned} |f|_{l,\mu} &= \sum_{j\leq l}\sup_{Y\geq 0} e^{\mu Y}\|\partial_\phi^{j} f(Y,\cdot)\|_{L^2} \\ &+ \sum_{0<j_2\leq 2}\sum_{j_1\leq l-2}\sup_{Y\geq 0} e^{\mu Y}\|\partial_\phi^{j_1}\partial_Y^{j_2} f(Y,\cdot)\|_{L^2}. \end{aligned} \tag{2.4}$$

We now introduce the dependence on time. We require differentiability with respect to time only up to the first order. This is typical of parabolic type equations in presence of a boundary. (see [4]).

Definition 2.6 $K_T^{l,\mu}$ *with* $\mu > 0$ *is the set of all functions* $f(Y,\phi,t)$ *such that*

- $f \in L^\infty([0,T], K^{l,\mu})$
- $\partial_t\partial_\phi^j f \in L^\infty([0,T], K^{0,\mu})$, *with* $j \leq l-2$

The norm is given by:

$$\begin{aligned} |f|_{l,\mu,T} &= \sum_{0<j_2\leq 2}\sum_{j_1\leq l-2}\sup_{0\leq t\leq T}\sup_{Y\geq 0} e^{\mu Y}\|\partial_\phi^{j_1}\partial_Y^{j_2} f(Y,\cdot,t)\|_{L^2} \\ &+ \sum_{j\leq l-2}\sup_{0\leq t\leq T}\sup_{Y\geq 0} e^{\mu Y}\|\partial_t\partial_\phi^{j} f(Y,\cdot,t)\|_{L^2}. \end{aligned} \tag{2.5}$$

We now introduce the ambient spaces for the error equation. All functions belonging to the following spaces are functions L^2 with respect to both tangential and normal variables. Notice that, due to the presence in the error equation of the rapidly varying terms arising from the Prandtl solution, the solution of the error equation will have a fast dependence on r. Therefore, in the following spaces, all the derivatives of order j with respect to r are weighted with ε^j.

Definition 2.7 L^l *is the set of all functions* $f(r,\phi)$ *such that*

- $r^{-j_1}\partial_\phi^{j_1}\varepsilon^{j_2}\partial_r^{j_2} f \in L^2_r$ *with* $j_2 \leq 2$ *and* $j_1 + j_2 \leq l$, *with* $j_1 \leq l-2$ *if* $j_2 > 0$.

The norm of $f \in L^l$ *is given by*

$$\|f\|_l = \sum_{j_1\leq l}\|r^{-j_1}\partial_\phi^{j_1} f\|_r + \sum_{0<j_2\leq 2}\sum_{j_1\leq l-2}\|r^{-j_1}\partial_\phi^{j_1}\varepsilon^{j_2}\partial_r^{j_2} f\|_r. \tag{2.6}$$

Definition 2.8 $L_T^{\prime l}$ *is the set of all functions* $f(\phi,t)$ *such that*

- $\partial_\phi^j f \in L^\infty([0,T], H^{\prime 0})$ *with* $j \leq l$.
- $\partial_\phi^j\partial_t f \in L^\infty([0,T], H^{\prime 0})$ *with* $j \leq l-2$.

The norm of $f \in L_T^{\prime l}$ *is given by*

$$\|f\|_{l,T} = \sum_{j\leq l}\sup_{0\leq t\leq T}\|\partial_\phi^j f(\cdot,t)\|_{L^2} + \sum_{j\leq l-2}\sup_{0\leq t\leq T}\|\partial_\phi^j\partial_t f(\cdot,t)\|_{L^2}\ . \tag{2.7}$$

Definition 2.9 *L^l_T is the set of all functions $f(r,\phi,t)$ such that:*

- $f \in L^\infty([0,T], L^l)$
- $r^{-j}\partial^j_\phi \partial_t f \in L^\infty([0,T], L^0)$ *with* $j \leq l-2$

The norm of $f \in L^l_T$ is given by:

$$\begin{aligned} \|f\|_{l,T} &= \sum_{0<j_2\leq 2} \sum_{j_1 \leq l-2} \sup_{0\leq t \leq T} \|r^{-j_1}\partial^{j_1}_\phi \varepsilon^{j_2} \partial^{j_2}_r f(\cdot,\cdot,t)\|_r \\ &+ \sum_{j\leq l-2} \sup_{0\leq t\leq T} \|r^{-j}\partial^j_\phi \partial_t f(\cdot,\cdot,t)\|_r. \end{aligned} \tag{2.8}$$

3 The Stokes equations: explicit formulas.

In this section we will give an explicit formula for the solution of the Stokes equations. In Subsection 3.1 we shall introduce some notations and a pseudo-differential operator that will be used in the last step of the solution of the Stokes systems. In Subsection 3.2 we will introduce the Weber transform and some of its properties. In Subsection 3.3 the Stokes equations are considered and the explicit solution is given through the Weber transform. Estimates are also given.

3.1 The Ukai operator

We will consider the Fourier expansion of $f(\phi)$:

$$f(\phi) = \sum_{-\infty}^{+\infty} e^{ik\phi} \hat{f}(k)$$

where

$$\hat{f}(k) = \int_0^{2\pi} e^{-ik\phi} f(\phi)\, d\phi.$$

In what follows we shall adopt the convention that k is the dual variable of the variable ϕ. If T is an operator acting on a function $f(\phi)$ such that:

$$\widehat{Tf}(k) = \sigma(T)(k)\hat{f}(k),$$

then $\sigma(T)$ is called the symbol of the pseudo-differential operator T. With an abuse of notation we shall adopt the convention of omitting the distinction between a function and its Fourier coefficient and between an operator and its symbol.

We introduce the operator $U[\cdot,\cdot]$ which acts on functions $f(r,k)$ with $r \in [R,\infty)$ and $g(k)$:

$$U[f,g] = U_b(g) + U_s(f)\ , \tag{3.1}$$

where:

$$U_b(g) = \left(\frac{R}{r}\right)^{|k|+1} g(k)\ . \tag{3.2}$$

$$U_s(f) = \frac{|k|}{r^{|k|+1}} \int_R^r dr' r'^{|k|} f(r',k)\ . \tag{3.3}$$

The operator $U[f,g]$ solves the following problem:

$$(r\partial_r + |k|+1)U[f,g] = |k|f, \tag{3.4}$$
$$\gamma_R U[f,g] = g(k). \tag{3.5}$$

We now give an estimate on the following operator that will be useful in the sequel.

Proposition 3.1 *Let $g \in H'^l$ and let $f \in L^l$. Then $U_b(g) \in H^l$ and $U_s(f) \in L^l$ and the following estimates hold:*

$$|U_b(g)|_l \le c|g|_l\ , \qquad \|U_s(f)\|_l \le c\|f\|_l.$$

Proof:
The proof of the first estimate is obvious since $\left(\frac{R}{r}\right)^{|k|+1} \le 1$. Let us consider the second inequality:

$$\begin{aligned}
\|U_s(f)\|_r^2 &= \sum_{k=-\infty}^{\infty} \int_R^\infty dr\, r \left| \left(\frac{1}{r}\right)^{|k|+1} |k| \int_R^r dr'\, r'^{|k|} f(r')\right|^2 \\
&= \sum_{k=-\infty}^{\infty} \int_R^\infty dr \left| \left(\frac{1}{r}\right)^{|k|+1/2} |k| \int_R^r dr'\, r'^{|k|} f(r')\right|^2 \\
&\le c \sum_{k=-\infty}^{\infty} \int_R^\infty dr\, r^{-(|k|+1/2)} |k| \int_R^r dr'\, r'^{|k|-1/2} \left|r'^{1/2} f(r')\right|^2 \\
&= c \sum_{k=-\infty}^{\infty} \int_R^\infty dr \frac{d}{dr}\left[-(\frac{|k|}{|k|-1/2})\, r^{-|k|+1/2}\right] \int_R^r dr'\, r'^{|k|-1/2} \left|r'^{1/2} f(r')\right|^2 \\
&\le c \sum_{k=-\infty}^{\infty} \Big\{ -\int_R^\infty dr \frac{d}{dr}\left[r^{-|k|+1/2} \int_R^r dr'\, r'^{|k|-1/2} \left| r'^{1/2} f(r')\right|^2\right] \\
&\qquad + \int_R^\infty dr\, r|f(r)|^2 \Big\} \\
&\le c\|f(r)\|_r^2.
\end{aligned}$$

In the above estimate we used Jensen's inequality in passing from the second to the third line. We used also the fact that $\frac{|k|}{|k|-1/2} \le c$.

3.2 The Weber transform and the heat operators

This subsection deals with the definition and the properties of the Weber transform which arises naturally in the discussion of axisymmetrical problems formulated in cylindrical polar coordinates (see [5]). Through the Weber transform we will solve the heat equation and define the heat operator that will be useful for the solution of the Stokes system.

Given $f(r)$ defined on $[R,\infty)$, the Weber transform of order ν is defined by the formula:

$$W_\nu\{f\}(p) = \int_R^\infty dr\, r\, Z_\nu(pr)\, f(r), \tag{3.6}$$

where

$$Z_\nu(pr) = J_\nu(pr)\, Y_\nu(pR) - Y_\nu(pr)\, J_\nu(pR).$$

In the above expression $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of order ν of the first and the second kind respectively, $Z_\nu(pr)$ are the cylinder functions satisfying the boundary condition $Z_\nu(pR) = 0$.

Using the following completeness relation:

$$\frac{\delta(x-x')}{x'} = \int_0^\infty \frac{Z_\nu(px)Z_\nu(px')}{J_\nu^2(pR)+Y_\nu^2(pR)}\, p\, dp, \tag{3.7}$$

one can get the inverse Weber transform, which is defined by:

$$f(r) = \int_0^\infty dp\, p\, \frac{Z_\nu(pr)}{J_\nu^2(pR)+Y_\nu^2(pR)}\, W_\nu\{f\}(p). \tag{3.8}$$

Now we will briefly recall some properties of the above transform that will be used in the following sections. The proofs are straightforward and make use of the fact that the cylinder functions satisfy Bessel differential equation, namely:

$$\frac{d}{dr}\left[r\, \frac{d}{dz} Z_\nu(pr) \right] + r\left(p^2 - \frac{\nu^2}{r^2}\right) Z_\nu(pr) = 0 \tag{3.9}$$

and the standard recurrence relation:

$$Z_\nu(pr) = \frac{pr}{2\nu}\,[\, Z_{\nu-1}(pr) + Z_{\nu+1}(pr)\,] \tag{3.10}$$

Lemma 3.1 *(Parseval's relation)*
If $f, g \in L_r^2$, then $W_\nu\{f\}, W_\nu\{g\} \in L_r^2$ and

$$\int_R^\infty dr\, r\, f(r)\, g(r) = \int_0^\infty dp\, p\, \frac{W_\nu\{f\}\, W_\nu\{g\}}{J_\nu^2(pR)+Y_\nu^2(pR)}.$$

Lemma 3.2 *(Bessel operator)*
If the Bessel operator of order ν is defined as

$$B_\nu(f) = \left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{\nu^2}{r^2}\right) f,$$

then

$$W_\nu\{B_\nu(f)\} = \frac{2}{\pi} f(R) - p^2 W_\nu\{f\}$$

provided that both $rf'(r)$ and $rf(r)$ decay as $r \to \infty$.

In the proof of the last proposition one uses the fact that the wronskian of J_ν and Y_ν, $\mathcal{W}[J_\nu, Y_\nu](x)$ is given by:

$$\mathcal{W}[J_\nu, Y_\nu](x) = J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x) = \frac{2}{\pi x}.$$

We are now ready to solve the heat equation expressed in polar coordinates on an exterior circular domain. It is known that the initial boundary value problem for the heat equation is uniquely solvable in Sobolev spaces, provided that the initial and the boundary data satisfy the compatibility conditions (see [4]). Since we are interested in giving an explicit solution formula for the below heat systems, we will make use of the Weber transform while we will refer to the cited bibliography for the estimates in the functional spaces.

Let us first consider the heat equation with source term, zero boundary data and zero initial data:

$$
\begin{aligned}
[\partial_t - \varepsilon^2(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi})]u(r,\phi,t) &= f(r,\phi,t)\ , && (3.11)\\
\gamma_R u &= 0\ , && (3.12)\\
u(r,\phi,t=0) &= 0\ . && (3.13)
\end{aligned}
$$

If we take the Fourier transform with respect to the ϕ variable, the system for the k-th mode is:

$$
\begin{aligned}
[\partial_t - \varepsilon^2(\partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2})]u(r,k,t) &= f(r,k,t)\ , && (3.14)\\
\gamma_R u &= 0\ , && (3.15)\\
u(r,k,t=0) &= 0\ . && (3.16)
\end{aligned}
$$

Taking the Weber transform of Eq. (3.14) and using the Lemma 3.2, we get the following ODE for $W_k\{u\}$:

$$
\begin{aligned}
\left(\frac{d}{dt} + \varepsilon^2 p^2\right) W_k\{u\}(p,k,t) &= W_k\{f\}, && (3.17)\\
W_k\{u\}(p,k,t=0) &= 0. && (3.18)
\end{aligned}
$$

whose solution is given by:

$$
W_k\{u\} = \int_0^t ds\, e^{-\varepsilon^2 p^2 (t-s)}\, W_k\{f\}, \tag{3.19}
$$

The solution u for the k-mode is then obtained by inversion of Eq. (3.19):

$$
u(r,k,t) = \mathcal{E}_k^{(2)}\, f \tag{3.20}
$$

where the heat operator $\mathcal{E}_k^{(2)}$ is defined by:

$$
\begin{aligned}
\mathcal{E}_k^{(2)}\, f &= \int_0^\infty dp\, p\, \frac{Z_k(pr)}{J_k^2(pR) + Y_k^2(pR)} \int_0^t ds\, e^{-\varepsilon^2 p^2 (t-s)}\, W_k\{f\}\\
&\equiv W_k^{-1}\left\{ \int_0^t ds\, e^{-\varepsilon^2 p^2 (t-s)}\, W_k\{f\} \right\}. && (3.21)
\end{aligned}
$$

The requested solution of Eqs. (3.11)-(3.13) is the Fourier series whose coefficients are the $u(r,k,t)$ given in (3.21).

The procedure to solve the homogeneous heat equation with initial data and zero boundary data is analogous. Given:

$$
\begin{aligned}
[\partial_t - \varepsilon^2(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi})]u &= 0, & (3.22)\\
\gamma_R u &= 0\,, & (3.23)\\
u(r,\phi,t=0) &= u_0(r,\phi)\,, & (3.24)
\end{aligned}
$$

one takes the Weber transform of the equation to get

$$
W_k\{u\}(p,k,t) = e^{-\varepsilon^2 p^2 t} W_k\{u_0\}. \tag{3.25}
$$

Taking the inverse Weber transform, one gets:

$$
u(r,k,t) = \mathcal{E}_k^{(0)}\, u_0 \tag{3.26}
$$

where the operator $\mathcal{E}_k^{(0)}$ is defined to be:

$$
\mathcal{E}_k^{(0)}\, u_0 = W_k^{-1}\left\{e^{-\varepsilon^2 p^2 t} W_k\{u_0\}\right\}. \tag{3.27}
$$

Analogously one can introduce the operator $\mathcal{E}_k^{(1)}$ which solves the homogeneous heat equation with boundary data and zero initial data, namely:

$$
\begin{aligned}
[\partial_t - \varepsilon^2(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi})]u &= 0, & (3.28)\\
\gamma_R u &= g(\phi,t)\,, & (3.29)\\
u(r,\phi,t=0) &= 0\,, & (3.30)
\end{aligned}
$$

Using again the Weber transform, the solution of Eqs. (3.28)-(3.30) can be written in the form:

$$
u = \mathcal{E}_k^{(1)}\, g \tag{3.31}
$$

where the operator $\mathcal{E}_k^{(1)}$ is expressed as:

$$
\mathcal{E}_k^{(1)}\, g = W_k^{-1}\left\{\int_0^t ds\, e^{-\varepsilon^2 p^2 (t-s)}\, \frac{2\varepsilon^2}{\pi} g\right\}. \tag{3.32}
$$

Finally we solve the heat equation with source term, boundary data and initial data:

$$[\partial_t - \varepsilon^2(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\phi\phi})]u = f(r,\phi,t) \quad (3.33)$$

$$\gamma_R u = g(\phi,t) , \quad (3.34)$$

$$u(r,\phi,t=0) = u_0(r,\phi) . \quad (3.35)$$

Then the solution of the system (3.33)-(3.35) can be written in the form:

$$u(r,\phi,t) = \mathcal{E}_k^{(2)}\, f + \mathcal{E}_k^{(1)}\, g + \mathcal{E}_k^{(0)}\, u_0 \quad (3.36)$$

We now give some estimates on the solutions of the above heat equations.

Lemma 3.3 *Let $f(r,\phi,t) \in L^l_T$. Then the solution u of the heat equations(3.11)-(3.13) is in L^l_T and the following estimate holds:*

$$\|u\|_{l,T} \leq c\, \|f\|_{l,T}.$$

Lemma 3.4 *Let $u_0(r,\phi) \in L^l$ and let the compatibility condition $\gamma_R u_0 = 0$ be satisfied. Then the solution u of the heat equations (3.22)-(3.24) is in L^l_T and the following estimate holds:*

$$\|u\|_{l,T} \leq c\, \|u_0\|_l.$$

Lemma 3.5 *Let $g(\phi,t) \in L^{\prime l}_T$ and let the compatibility condition $g(\phi,t=0) = 0$ be satisfied. Then the solution u of the heat equations (3.28)-(3.30) is in L^l_T and the following estimate holds:*

$$\|u\|_{l,T} \leq c\ \ \|g\|_{l,T}.$$

Proposition 3.2 *Let $f(r,\phi,t) \in L^l_T$, $u_0(r,\phi) \in L^l$ and $g(\phi,t) \in L^{\prime l}_T$. Let the compatibility condition $\gamma_R u_0 = g(\phi,t=0)$ be satisfied. Then the solution u of the heat equations (3.33)-(3.35) is in L^l_T and the following estimate holds:*

$$\|u\|_{l,T} \leq c\ [\ \|f\|_{l,T} + \|u_0\|_l + \|g\|_{l,T}] .$$

One can find the proofs of the above Lemmas and Proposition in [4], Chapter IV. Notice that in all of the above statements the constant c does not depend on ε.

3.3 The Stokes equation

In this subsection we consider the following equations:

$$\partial_t u_\phi - \varepsilon^2[(\partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2})u_\phi - r^{-2}u_\phi + \frac{2ik}{r^2}u_r] + r^{-1}\partial_\phi p = f_\phi(r,k,t) , \quad (3.37)$$

$$\partial_t u_r - \varepsilon^2[(\partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2})u_r - r^{-2}u_r - \frac{2ik}{r^2}u_\phi] + \partial_r p = f_r(r,k,t) , \quad (3.38)$$

$$\nabla \cdot \boldsymbol{u} = 0 , \quad (3.39)$$

$$\gamma_R \boldsymbol{u} = \boldsymbol{g}(k,t) , \quad (3.40)$$

$$\boldsymbol{u}(r,\phi,t=0) = \boldsymbol{u}_0 . \quad (3.41)$$

Taking the divergence of Eqs. (3.37)-(3.38) a straightforward calculation shows that the pressure p is harmonic, i.e. using the Fourier transform:

$$\Delta p = (\frac{1}{r}\partial_r - \frac{|k|}{r^2})(r\partial_r + |k|)p = 0. \tag{3.42}$$

Imposing the pressure to be finite at infinity we get:

$$(r\partial_r + |k|)p = 0$$

which also implies:

$$(r\partial_r + |k| + 1)\ \partial_r p = \partial_r\ (r\partial_r + |k|)\ p = 0. \tag{3.43}$$

On the other hand, using the incompressibility condition, Eq. (3.38) becomes:

$$\partial_t u_r - \varepsilon^2 \Delta_M u_r + \partial_r p = f_r\ . \tag{3.44}$$

where $\Delta_M \equiv (\partial_{rr} + \frac{3}{r}\partial_r - \frac{k^2-1}{r^2})$. It is easily seen that Δ_M can be written in the form:

$$\Delta_M = (\frac{1}{r}\partial_r - \frac{|k|-1}{r^2})(r\partial_r + |k| + 1).$$

Therefore, if we define $|k|\tau$ as

$$|k|\tau = (r\partial_r + |k| + 1)u_r \tag{3.45}$$

and apply $(r\partial_r + |k| + 1)$ to Eq. (3.44), we get the following equation for $|k|\tau$:

$$\partial_t |k|\tau - \varepsilon^2 (r\partial_r + |k| + 1)(\frac{1}{r}\partial_r - \frac{|k|-1}{r^2})\ |k|\tau = (r\partial_r + |k| + 1)f_r.$$

Notice that the pressure does not appear because of Eq. (3.43). Moreover, since

$$(r\partial_r + |k| + 1)(\frac{1}{r}\partial_r - \frac{|k|-1}{r^2}) = \partial_{rr} + \frac{1}{r}\partial_r - \frac{(|k|-1)^2}{r^2}$$

we get the following equation for $|k|\tau$:

$$\left[\partial_t - \varepsilon^2\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{(|k|-1)^2}{r^2}\right)\right]|k|\tau = (r\partial_r + |k| + 1)f_r. \tag{3.46}$$

The boundary and initial condition read:

$$\gamma|k|\tau = \gamma(r\partial_r + |k| + 1)u_r = \gamma(-iku_\phi - u_r + (|k|+1)u_r) \ = \ |k|V_1\boldsymbol{g}, \tag{3.47}$$

$$|k|\tau\ (t=0) \ = \ |k|V_1\boldsymbol{u}_0, \tag{3.48}$$

where the operator V_1 is defined as:

$$V_1 \boldsymbol{g} = g_r - N' g_\phi$$

and N' is defined as:

$$N' = \frac{ik}{|k|}.$$

Notice that, using the incompressibility condition for $\boldsymbol{f}$, the source term in (3.46) can be expressed in terms of V_1 as $|k| V_1 \boldsymbol{f}$. Since the spatial operator on the left hand side of Eq. (3.46) is the Bessel operator of order $|k|-1$, to solve Eqs. (3.46)-(3.48) we will make use of the Weber transform. Using the result given in Eq. (3.36), one gets:

$$\tau(r,k,t) = \mathcal{E}^{(2)}_{|k|-1}\, V_1 \boldsymbol{f} + \mathcal{E}^{(1)}_{|k|-1}\, V_1 \boldsymbol{g} + \mathcal{E}^{(0)}_{|k|-1}\, V_1 \boldsymbol{u}_0 \,. \tag{3.49}$$

Solving equation (3.45) with the boundary condition (3.40) one gets for u_r:

$$u_r = U\left[\, \tau, g_r \,\right], \tag{3.50}$$

where U is the operator defined in the Subsection 3.1. Through the incompressibility condition one gets u_ϕ:

$$u_\phi = N' \left[\tau(r,k,t) - u_r(r,k,t) \right] \,. \tag{3.51}$$

Using the expressions (3.50) and (3.51) one can finally express the solution of the Stokes equations in the following form:

$$\boldsymbol{u} = \boldsymbol{u}_{(b)} + \boldsymbol{u}_{(s)} \,, \tag{3.52}$$

where

$$\begin{aligned} \boldsymbol{u}_{(b)} &= \left(U_b(g_r), -N' U_b(g_r) \right) \,, & (3.53) \\ \boldsymbol{u}_{(s)} &= \left(U_s(\tau), N'(\tau - U_s(\tau)) \right) \,. & (3.54) \end{aligned}$$

We now estimate the above solution. We first state the following Lemma:

Lemma 3.6 *Let $\boldsymbol{u}_0 \in L^l$, $\boldsymbol{f} \in L^l_T$ and $\boldsymbol{g} \in L'^l_T$. Suppose the compatibility condition between the boundary and initial data $\gamma_R \boldsymbol{u}_0 = \boldsymbol{g}(\phi, t=0)$ is satisfied. Then $\tau \in L^l_T$ and the following estimate holds:*

$$\|\tau\|_{l,T} \le c \left(\|\boldsymbol{u}_0\|_l + \|\boldsymbol{f}\|_{l,T} + \|\boldsymbol{g}\|_{l,T} \right) \,. \tag{3.55}$$

The proof of the above Lemma is based on the fact that τ satisfies Eqs. (3.46)-(3.48) which are of the same form as (3.33)-(3.35), with the boundary and initial data satisfying the compatibility conditions. Therefore Proposition 3.2 applies and the statement of the Lemma follows.

The main result of this Section is the following Proposition:

Proposition 3.3 *Let $\boldsymbol{u}_0 \in L^l$, $\boldsymbol{f} \in L^l_T$ and $\boldsymbol{g} \in L'^l_T$. Suppose the compatibility condition between the boundary and initial data $\gamma_R \boldsymbol{u}_0 = \boldsymbol{g}(\phi, t=0)$ is satisfied. Then the solution of Eqs. (3.37)-(3.41) $\boldsymbol{u}$ can be decomposed in the form (3.52), with $\boldsymbol{u}_{(b)} \in H^l_T$, $\boldsymbol{u}_{(s)} \in L^l_T$ and the following estimates hold:*

$$
\begin{aligned}
|\boldsymbol{u}_{(b)}|_{l,T} &\leq c\,\|g\|_{l,T} \\
\|\boldsymbol{u}_{(s)}\|_{l,T} &\leq c\,[\|\boldsymbol{u}_0\|_l + \|\boldsymbol{f}\|_{l,T} + \|\boldsymbol{g}\|_{l,T}]\,.
\end{aligned}
$$

The proof is easily achieved using the explicit expressions for $\boldsymbol{u}_{(b)}$ and $\boldsymbol{u}_{(s)}$ given in Eqs. (3.53) and (3.54), and through the estimate on τ given in Lemma 3.6, and through the estimates on the operators U_b and U_s given in Proposition 3.1.

4 The Stokes equations: asymptotic analysis.

In this section we shall analyze Eqs. (1.1)-(1.5) in the limit of small viscosity.

We impose an initial condition of the form:

$$
\boldsymbol{u}_0 = \boldsymbol{u}_0^E + \boldsymbol{u}_0^P + \varepsilon \boldsymbol{w}_0 \tag{4.1}
$$

with

$$
\begin{aligned}
\boldsymbol{u}_0^E &\in H^l, && (4.2)\\
u_{0\phi}^P &\in K^{l,\mu}, && (4.3)\\
\boldsymbol{w}_0 &\in L^l, && (4.4)
\end{aligned}
$$

and $\boldsymbol{u}_0^P = (\varepsilon u_{0r}^P, u_{0\phi}^P)$ where the radial component u_{0r}^P is determined through the incompressibility condition, see Eq. (4.9) below. The initial conditions satisfy the following compatibility conditions:

$$
\begin{aligned}
\gamma_R u_{0r}^E = g_r(t=0) &= 0, && (4.5)\\
\gamma_R u_{0\phi}^P &= -\gamma_R u_{0\phi}^E + g_\phi(t=0), && (4.6)\\
\gamma_R \boldsymbol{w}_0 &= (-\gamma_R u_{0r}^P, 0)\,. && (4.7)
\end{aligned}
$$

Moreover the initial conditions satisfy the incompressibility conditions:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot \boldsymbol{u}_0^E &= 0\,, && (4.8)\\
u_{0r}^P &= \frac{1}{R+\varepsilon Y}\int_Y^\infty dY'\partial_\phi u_{0\phi}^P\,, && (4.9)\\
\boldsymbol{\nabla}\cdot \boldsymbol{w}_0 &= 0\,. && (4.10)
\end{aligned}
$$

We look for a solution of Eqs. (1.1)-(1.5) of the form:

$$
u_r = u_r^E + \varepsilon u_r^P + \varepsilon w_r, \quad u_\phi = u_\phi^E + u_\phi^P + \varepsilon w_\phi, \quad p = p^E + \varepsilon p^w,
$$

where $\boldsymbol{u}^E$, $\boldsymbol{u}^P$ and $\boldsymbol{w}$ are the inviscid solution, the boundary layer solution and the correction term respectively and satisfy the following systems:

$$
\begin{aligned}
\partial_t \boldsymbol{u}^E + \boldsymbol{\nabla} p^E &= 0, && (4.11)\\
\boldsymbol{\nabla}\cdot \boldsymbol{u}^E &= 0, && (4.12)\\
\gamma_R u_r^E &= g_r(k,t), && (4.13)\\
\boldsymbol{u}^E(r,k,t=0) &= \boldsymbol{u}_0^E. && (4.14)
\end{aligned}
$$

$$
\begin{aligned}
(\partial_t - \partial_{YY} + \frac{\varepsilon^2 k^2}{R^2}) u_\phi^P &= 0 \,, & (4.15)\\
\gamma u_\phi^P &= -\gamma_R u_\phi^E + g_\phi \,, & (4.16)\\
u_\phi^P(Y \to \infty, \phi, t) &= 0 \,, & (4.17)\\
u_\phi^P(Y, \phi, t=0) &= u_{0\phi}^P \,. & (4.18)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t w_\phi - \varepsilon^2 \left[\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2} \right) w_\phi - \frac{1}{r^2} w_\phi + \frac{2ik}{r^2} w_r \right] + \frac{1}{r}\partial_\phi p^w &= f_\phi \,, & (4.19)\\
\partial_t w_r - \varepsilon^2 \left[\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2} \right) w_r - \frac{1}{r^2} w_r - \frac{2ik}{r^2} w_\phi \right] + \partial_r p^w &= f_r \,, & (4.20)\\
\nabla \cdot \boldsymbol{w} &= 0 \,, & (4.21)\\
\gamma_R \boldsymbol{w} &= (-\gamma u_r^P, 0) \,, & (4.22)\\
\boldsymbol{w}(r,k,t=0) &= \boldsymbol{w}_0 \,, & (4.23)
\end{aligned}
$$

where the expressions of the source terms $(f_r(r,k,t); f_\phi(r,k,t))$ are given by:

$$
f_r = \varepsilon \left(-\frac{ik}{r^2} u_\phi^P + \Delta u_{0r}^E - \frac{u_{0r}^E}{r^2} - \frac{2ik}{r^2} u_{0\phi}^E \right) + \varepsilon^2 \frac{k^2}{R^2} \frac{r^2 - R^2}{r^2} u_r^P \,, \tag{4.24}
$$

$$
\begin{aligned}
f_\phi = \varepsilon & \left[\frac{1}{r}\partial_r u_\phi^P - \frac{u_\phi^P}{r^2} \left(1 - \frac{k^2}{R^2}(r^2 - R^2) \right) + \Delta u_{0\phi}^E - \frac{u_{0\phi}^E}{r^2} + \frac{2ik}{r^2} u_{0r}^E \right] \\
& + \varepsilon^2 \, \frac{2ik}{r^2} \, u_r^P \,.
\end{aligned} \tag{4.25}
$$

The Euler equations (4.11)-(4.14) have been obtained from Eqs. (1.1)-(1.5) neglecting the viscosity term and imposing the boundary condition only for the radial component. Prandtl equations are obtained writing Eqs. (1.1)-(1.5) in terms of the rescaled radial variable $Y = (r - R)/\varepsilon$ and imposing the usual boundary layer approximation. The boundary condition (4.16) ensures the right value at the boundary for $\boldsymbol{u}$. The radial velocity εu_r, which is $O(\varepsilon)$, can be computed from the incompressibility condition. Notice that in writing the expressions (4.24) and (4.25) for the source term we have used the fact that $\Delta \boldsymbol{u}^E = \Delta \boldsymbol{u}_0^E$: we shall prove this in the following subsection, see Eq. (4.28) below.

We now solve the above equations.

4.1 Euler equations

The solution of Eqs. (4.11)-(4.14) is given by:

$$
\boldsymbol{u}^E = \boldsymbol{u}_0^E + \nabla N g_r, \tag{4.26}
$$

where the operator N, defined by:

$$
N = -\frac{R}{|k|} \left(\frac{R}{r} \right)^{|k|} , \tag{4.27}
$$

solves the Laplace equation with Neumann boundary condition, i.e.

$$\begin{aligned} \Delta N g_r &= 0, \\ \gamma_R \partial_r N g_r &= g_r. \end{aligned}$$

The expression of the pressure is given by:

$$p^E = N \partial_t g_r.$$

From Eq. (4.26) it is clear that

$$\Delta \boldsymbol{u}^E = \Delta \boldsymbol{u}_0^E. \tag{4.28}$$

4.2 Prandtl equations

Let us first introduce the heat operators in the half plane. The heat kernel $E_0(k, Y, t)$ is defined by:

$$E_0(k, Y, t) = \frac{1}{(4\pi t)^{1/2}} \, e^{-\frac{\varepsilon^2 k^2}{R^2} t} \, e^{-\frac{Y^2}{4t}}.$$

The convolution between the heat kernel and the odd extension (to $Y < 0$) of the function f is the operator $E_0(t)$, namely:

$$E_0(t) f = \int_0^\infty dY' \left[E_0(k, Y - Y', t) - E_0(k, Y + Y', t) \right] f(Y').$$

It solves the heat equation on the half plane $Y > 0$ with initial data f and with zero boundary data.

The operator E_1 solves the heat equation with boundary data g and zero initial data and is given by:

$$E_1 g(t) = \int_0^t ds \; e^{-\frac{\varepsilon^2 k^2}{R^2}(t-s)} \, \frac{Y}{t-s} \frac{e^{-Y^2/4(t-s)}}{\sqrt{4\pi (t-s)}} \, g(s).$$

If one introduces the operator M, which acts on vector functions and is defined as :

$$M \boldsymbol{g} = g_\phi + N' g_r,$$

and using Eq. (4.26) one can write the boundary condition for u_ϕ^P, Eq. (4.16), in the form:

$$\gamma u_\phi^P = M \boldsymbol{g} - \gamma_R u_{0\phi}^E \,. \tag{4.29}$$

Hence the solution of boundary layer equation (4.15) with the boundary conditions Eq. (4.29) and Eq. (4.17), and the initial condition Eq. (4.18) is given by:

$$u_\phi^P = E_0(t) u_{0\phi}^P + E_1 \left[M \boldsymbol{g} - \gamma_R u_{0\phi}^E \right] \,. \tag{4.30}$$

The expression for the radial component u_r^P is obtained from the incompressibility condition:

$$u_r^P = \frac{1}{\varepsilon Y + R} \int_Y^\infty dY' \partial_\phi u_\phi^P \,. \tag{4.31}$$

4.3 The correction term

The equations (4.19)-(4.23) for the correction $\boldsymbol{w}$ are of the same form as Eqs. (3.37)-(3.41). Therefore the solution is explicitly given by Eqs. (3.50) and (3.51):

$$w_r = U\left[\tau'(r,k,t), |k|N'\beta\right] , \tag{4.32}$$
$$w_\phi = N'\left[\tau'(r,k,t) - w_r(r,k,t)\right] , \tag{4.33}$$

where we have defined $\tau'(r,k,t)$ as:

$$\tau'(r,k,t) = \mathcal{E}^{(2)}_{k-1}\, V_1 \boldsymbol{f} + \mathcal{E}^{(1)}_{k-1}\, |k|N'\beta + \mathcal{E}^{(0)}_{k-1}\, V_1 \boldsymbol{w}_0 \, . \tag{4.34}$$

In the above expression $\boldsymbol{f}$ is given by (4.24) and (4.25), while β is defined as:

$$\beta = -\frac{1}{R}\int_0^\infty u^P_\phi(Y',k,t)\, dY' \, . \tag{4.35}$$

To give a better estimate on the error it is useful to decompose $V_1\boldsymbol{f}(r,k,t)$ into the sum of two terms:

$$V_1\boldsymbol{f}(r,k,t) = \mathcal{F}^{l-1}(r,k,t) + \varepsilon\, \mathcal{F}^{l-2}(r,k,t) \, ,$$

where

$$\begin{aligned} \mathcal{F}^{l-1}(r,k,t) &= \varepsilon\left\{\frac{1}{r^2}\left(-ik+N'\right)u^P_\phi - \frac{N'}{r}\partial_r u^P_\phi - \frac{1}{r^2}(u^E_{0r} - N'u^E_{0\phi}) \right. \\ &\quad \left. -\frac{2ik}{r^2}(u^E_{0\phi} + N'u^E_{0r})\right\} , \end{aligned} \tag{4.36}$$

$$\begin{aligned} \mathcal{F}^{l-2}(r,k,t) &= N'\left(\frac{1}{r^2} - \frac{1}{R^2}\right) k^2 u^P_\phi + \Delta(u^E_{0r} - N'u^E_{0\phi}) \\ &+ \varepsilon u^P_r\left(\frac{2|k|}{r^2} + \frac{k^2}{R^2}\frac{r^2-R^2}{r^2}\right) . \end{aligned} \tag{4.37}$$

The reason of the above decomposition is that, as we shall see in the following section, $\mathcal{F}^{l-1}$ and $\mathcal{F}^{l-2}$ give rise to two different contribution to $\varepsilon\boldsymbol{w}$. The term originating from $\mathcal{F}^{l-1}$ will be shown to belong to L^{l-1}_T with norm $O(\varepsilon)$. The term originating from $\mathcal{F}^{l-2}$ will have norm $O(\varepsilon^2)$ in L^{l-2}_T.

The above decoposition for $V_1\boldsymbol{f}$ gives rise to an analogous decomposition for τ' as given in (4.34). In fact one can write:

$$\tau' = \tau'^{l-1} + \varepsilon\tau'^{l-2} \, , \tag{4.38}$$

where

$$\tau'^{l-1}(r,k,t) = \mathcal{E}^{(2)}_{k-1}\, \mathcal{F}^{l-1} + \mathcal{E}^{(1)}_{k-1}\, |k|N'\beta + \mathcal{E}^{(0)}_{k-1}\, V_1\boldsymbol{w}_0 \tag{4.39}$$
$$\tau'^{l-2}(r,k,t) = \mathcal{E}^{(2)}_{k-1}\, \mathcal{F}^{l-2} \, . \tag{4.40}$$

The decomposition (4.38) for τ' allows one to see that the correction $\boldsymbol{w}$, as given by (4.32) and (4.33) has the following structure:

$$\boldsymbol{w} = \boldsymbol{w}^E + \boldsymbol{w}^{l-1} + \varepsilon\boldsymbol{w}^{l-2} \, , \tag{4.41}$$

where

$$w_r^E = U_b\left(|k|N'\beta\right) \ , \qquad w_\phi^E = -N'U_b\left(|k|N'\beta\right) \ ; \tag{4.42}$$

$$w_r^{l-1} = U_s\left(\tau'^{l-1}\right), \qquad w_\phi^{l-1} = N'\left(\tau'^{l-1} - U_s(\tau'^{l-1})\right) \ ; \tag{4.43}$$

$$w_r^{l-2} = U_s\left(\tau'^{l-2}\right) \ , \qquad w_\phi^{l-2} = N'\left(\tau'^{l-2} - U_s(\tau^{l-2})\right) \ . \tag{4.44}$$

We remind that the operators U_b and U_s have been introduced in (3.2) and (3.3) respectively.

4.4 Estimates

In this section we give the estimates on the inviscid term $\boldsymbol{u}^E$, on the boundary layer term $\boldsymbol{u}^P$ and on the correction term $\boldsymbol{w}$. For sake of notational simplicity we introduce the norm $\|\!| \cdot |\!\|_{l,\mu}$. Suppose $\boldsymbol{u}_0$ has the structure given in (4.1) with $\boldsymbol{u}_0^E \in H^l$, $u_{0\phi}^P \in K^{l,\mu}$ and $\boldsymbol{w}_0 \in L^l$. Then we define $\|\!|\boldsymbol{u}_0|\!\|_{l,\mu}$ as:

$$\|\!|\boldsymbol{u}_0|\!\|_{l,\mu} = |\boldsymbol{u}_0^E|_l + |u_{0\phi}^P|_{l,\mu} + \|\boldsymbol{w}_0\|_l .$$

We begin with the inviscid part $\boldsymbol{u}^E$, solution of Eqs. (4.11)-(4.14). First we state an estimate on the operator N as defined in (4.27).

Lemma 4.1 *Let $\psi \in H_T^{\prime l}$. Then $\boldsymbol{\nabla} N\psi \in H_T^l$, and the following estimate holds:*

$$|\boldsymbol{\nabla} N\psi|_{l,T} \leq c|\psi|_{l,T} \ .$$

The proof is obvious and is based on the representation (4.27) for the operator N and on the fact that $(R/r)^{|k|} \leq 1$.

Proposition 4.1 *Let $\boldsymbol{g} \in H_T^{\prime l}$ and let $\boldsymbol{u}_0$ satisfy (4.1)-(4.10). Then $\boldsymbol{u}^E \in H_T^l$ and the following estimate holds:*

$$|\boldsymbol{u}^E|_{l,T} \leq c\ \left(\|\!|\boldsymbol{u}_0|\!\|_{l,\mu} + |\boldsymbol{g}|_{l,T}\right) .$$

The proof is based on the explicit representation (4.26) for $\boldsymbol{u}^E$, and on Lemma 4.1.

We now pass to the boundary layer solution $\boldsymbol{u}^P$, solution of Eqs. (4.15)-(4.18).

Proposition 4.2 *Let $\boldsymbol{g} \in H_T^{\prime l}$ and let $\boldsymbol{u}_0$ satisfy (4.1)-(4.10). Then $u_\phi^P \in K_T^{l,\mu}$, $u_r^P \in K_T^{l-1,\mu}$, and the following estimate holds:*

$$\begin{aligned} |u_\phi^P|_{l,\mu,T} &\leq c\left(\|\!|\boldsymbol{u}_0|\!\|_{l,\mu} + |\boldsymbol{g}|_{l,T}\right) , \\ |u_r^P|_{l-1,\mu,T} &\leq c\left(\|\!|\boldsymbol{u}_0|\!\|_{l,\mu} + |\boldsymbol{g}|_{l,T}\right) . \end{aligned}$$

The proof of the above proposition is based on the explicit representation of u_ϕ^P given in (4.30) on the fact that $\gamma u_{0\phi}^P = \left[M\boldsymbol{g} - \gamma_R u_{0\phi}^E\right]_{t=0}$ (due to the compatibility conditions), and on the usual estimates on the heat operators given e.g. in [1] and [4].

Notice the loss of regularity (one derivative) in the radial component due to the incompressibility condition.

We now pass to the correction term, solution of Eqs. (4.19)-(4.23). We first state the following preliminary Lemmas.

Lemma 4.2 *Let $\boldsymbol{g} \in H_T^{\prime l}$, $\boldsymbol{u}_0$ satisfy (4.1)-(4.10), and let β be given by (4.35). Then $\beta \in H_T^{\prime l}$ and the following estimate holds:*

$$|\beta|_{l,T} \le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) .$$

The above Lemma is obvious and is based on the fact that $u_\phi^P \in K_T^{l,\mu}$ and therefore it is decaying (exponentially) in Y.

Lemma 4.3 *Let $\boldsymbol{g} \in H_T^{\prime l}$, $\boldsymbol{u}_0$ satisfy (4.1)-(4.10), and let $\tau^{\prime l-1}$ be given by (4.39). Then $\tau^{\prime l-1} \in L_T^{l-1}$ and the following estimate holds:*

$$|\tau^{\prime l-1}|_{l-1,T} \le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) .$$

Lemma 4.4 *Let $\boldsymbol{g} \in H_T^{\prime l}$, $\boldsymbol{u}_0$ satisfy (4.1)-(4.10), and let $\tau^{\prime l-2}$ be given by (4.40). Then $\tau^{\prime l-2} \in L_T^{l-2}$ and the following estimate holds:*

$$|\tau^{\prime l-2}|_{l-2,T} \le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) .$$

The proof of Lemma 4.3 is based on the fact that $\mathcal{F}^{l-1} \in L_T^{l-1}$, as can be seen by a direct inspection of (4.36), on the fact that $\gamma_R V_1 \boldsymbol{w}_0 = N'\beta|_{t=0}$ (compatibility conditions), and on Proposition 3.2.

The proof of Lemma 4.4 is based on the fact that $\mathcal{F}^{l-2} \in L_T^{l-2}$, as can be seen by a direct inspection of (4.37).

With the above Lemmas the following estimate on the correction $\boldsymbol{w}$ is obvious.

Proposition 4.3 *Let $\boldsymbol{g} \in H_T^{\prime l}$ and let $\boldsymbol{u}_0$ satisfy (4.1)-(4.10). Then $\boldsymbol{w}$ can be decomposed as in (4.41), with $\boldsymbol{w}^E \in H_T^{l-1}$, $\boldsymbol{w}^{l-1} \in L_T^{l-1}$, $\boldsymbol{w}^{l-2} \in L_T^{l-2}$, and the following estimates hold:*

$$\begin{aligned} |\boldsymbol{w}^E|_{l-1,T} &\le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) , \\ \|\boldsymbol{w}^{l-1}\|_{l-1,T} &\le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) , \\ \|\boldsymbol{w}^{l-2}\|_{l-2,T} &\le c\left(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}\right) . \end{aligned}$$

4.5 The main result

We summarize our results in the following Theorem:

Theorem 4.1 *Let $\boldsymbol{g} \in H_T^{\prime l}$ and let $\boldsymbol{u}_0$ satisfy (4.1)-(4.10). Then the solution of Eqs. (1.1)-(1.5) is of the form:*

$$u_r = u_r^E + \varepsilon u_r^P + \varepsilon w_r^E + \varepsilon w_r^{l-1} + \varepsilon^2 w_r^{l-2}, \quad u_\phi = u_\phi^E + u_\phi^P + \varepsilon w_\phi^E + \varepsilon w_\phi^{l-1} + \varepsilon^2 w_\phi^{l-2}, \quad p = p^E + \varepsilon p^w,$$

where $\boldsymbol{u}^E \in H_T^l$, $u_\phi^P \in K^{l,\mu,T}$, $u_r^P \in K^{l-1,\mu,T}$, $\boldsymbol{w}^E \in H_T^{l-1}$, $\boldsymbol{w}^{l-1} \in L_T^{l-1}$, $\boldsymbol{w}^{l-2} \in L_T^{l-2}$.

The following estimates hold:

$$\begin{aligned}
|\boldsymbol{u}^E|_{l,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ , \\
|u_\phi^P|_{l,\mu,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ , \\
|u_r^P|_{l-1,\mu,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ , \\
|\boldsymbol{w}^E|_{l-1,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ , \\
\|\boldsymbol{w}^{l-1}\|_{l-1,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ , \\
\|\boldsymbol{w}^{l-2}\|_{l-2,T} &\leq c\,(|\boldsymbol{g}|_{l,T} + \|\boldsymbol{u}_0\|_{l,\mu}) \ .
\end{aligned}$$

5 Concluding remarks

In this paper we have considered the zero viscosity limit of the time-dependent Stokes equations in the exterior of a disk. We proved that away from the boundary the solution of the Stokes equations converges to the solution of the linearized Euler equations. Close to the boundary instead, the solution has the structure of a boundary layer whose size is the square root of the viscosity. Several related problems suggest themselves. One could ask if the same results apply in the case of the full Navier-Stokes equations. In this case we believe that the additional hypothesis of analyticity of the initial data is necessary to prevent the appearance of a singularity in the solution of Prandtl equations. It would be also interesting the analysis of the Navier-Stokes equations linearized around a background flow (Oseen equations). In fact the presence of an inflection point in the background flow, with the appearance of instability, could destroy the regular asymptotic structure of the Stokes equations. These topics are under current investigations, and will be the subject of a forthcoming paper.

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