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December 1999

CAM Report 99-37

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http://www.math.ucla.edu/applied/cam/index.html

MOTIONS IN A BOSE CONDENSATE VII. BOUNDARY LAYER SEPARATION

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December 14, 1999

Abstract

The Bose condensate model is used to analyze the superfluid flow around an ion (modeled as a solid sphere) and to elucidate the mechanism of vortex ring emission from the sphere that occurs if its velocity exceeds a critical value. An asymptotic expansion is developed for the steady subcritical flow, using the ratio of the healing length to the radius of the sphere as a small parameter. This expansion allows for the compressibility of the condensate, and converges well enough for the critical ion velocity to be calculated accurately. The flow for supercritical ion velocities is computed numerically. Particular attention is paid to the question of where and why the vortex rings are emitted at a preferred location on the sphere's surface.

1 Introduction

This is the seventh in a series of papers devoted to the Bose condensate as applied to superfluid helium and especially superfluid vortices; see Roberts and Grant (1971), Grant (1973), Grant and Roberts (1974), Jones and Roberts (1982), Jones, Putterman and Roberts (1986), and Berloff and Roberts (1999).

Vortex nucleation by an impurity such as a positive ion ${}^4\mathrm{He}_2^+$ moving in superfluid helium at low temperature has been studied experimentally and theoretically (see, e.g. Donnelly, 1991), and has uncovered some interesting physics. The flow round an ion that is moving with a sufficiently small velocity, v, is well represented by one of the classical solutions of fluid mechanics, namely the flow of an inviscid incompressible fluid around a sphere. In this solution, the maximum flow velocity, $\mathbf{u}$, relative to the sphere is $3v/2$, and occurs on the equator of the sphere (defined with respect to the direction of motion of the sphere as polar axis). Above some critical velocity, v_c, the ideal superflow around the ion breaks down, leading to the creation of a vortex ring (Rayfield and Reif, 1964). The critical velocity can be roughly estimated by arguing that the vortex will be nucleated from the point where the relative velocity of ion and superfluid is greatest (the equator), and will occur when that velocity reaches the Landau critical velocity, v_L. If, using the incompressible model, we estimate the relative velocity as $3v/2$, we find that $v_c \approx 2v_L/3$, in rough agreement with experiment; see Table 8.2 of Donnelly (1991). Because $2v_L/3$ is only about 15% of the speed of sound, c, it appears that the incompressible model should perform reasonable well, and that an allowance for the compressibility of the superfluid is not a high priority. This was the basis of the original paper by Strayer *et al.* (1971) and the later developments of Muirhead *et al.* (1984), who created a theory of vortex nucleation that allowed them to calculate v_c, the form of the potential barrier that must be overcome for the creation of vortices both as encircling rings and vortex loops, and the nucleation rate. These calculations were carried out for a smooth rigid sphere moving through an ideal incompressible fluid.

The Bose condensate offers a different insight into the nucleation process. The condensate is a weakly interacting Bose gas that, in the Hartree approximation, is governed by an equation for the single particle wavefunction $\psi(\mathbf{x}, t)$ that was first derived by Ginsburg and Pitaevskii (1958) and Gross (1963); see (1) below. Using this equation, Grant and Roberts (1974) studied the negative ion (the electron bubble) and a positive ion, modeling the latter as a spherical, infinite potential barrier, on the surface of which ψ vanishes. Their solutions were derived by expansion in v/c, so that their leading order flow is incompressible. They did not observe vortex nucleation.

As a model of superfluidity, the condensate suffers from the defect that its dispersion relation does not possess a roton minimum, so that $v_L = c$. To observe vortex nucleation therefore, Grant and Roberts (1974) would have had to develop expansions appropriate for a compressible flow in which $\mathbf{u} = \mathrm{O}(c)$, which they did not do (although we do so in §3 below). It is possible to make the condensate model more realistic by replacing the $\delta-$function interaction potential between atoms, on which it is based, by a nonlocal potential. This restores the roton minimum and a realistic v_L but only at the expense of considerable complexity; see Berloff (1999). As for most recent research on our topic (e.g., Frisch *et al.*, 1992; Winiecki *et al.*, 1999), we shall employ the condensate model in its original form.

An important scale defined by the condensate model is the 'healing length', a, defined in (10) below. This determines the radius of a vortex core and the thickness of the 'healing layer' that forms at a potential barrier (such as the ion surface in our model). The radius, b, of the ion is large compared with a, and asymptotic solutions for $\epsilon \equiv a/b \to 0$ become relevant; see §3. Such a solution has two parts, an interior or 'boundary layer' structure that matches smoothly to an exterior or 'mainstream' flow. In the mainstream, quantum effects are negligible at leading order, and the condensate becomes effectively a compressible inviscid fluid obeying the simple equation of state, $p \propto \rho^2$, where p is pressure and ρ is density; see (8) below.

There is some similarity between the flow of the condensate past the ion and the motion of a viscous fluid past a sphere at large Reynolds numbers, the healing layer being the counterpart of the viscous boundary layer. There are, however, important differences. At subcritical velocities, the flow of the condensate is symmetric fore and aft of the direction of motion, and the sphere experiences no drag. In contrast, the viscous boundary layer separates from the sphere, so evading D'Alembert's paradox, destroying the fore and aft symmetry, and therefore bringing about a drag on the sphere. Moreover, when $v > v_c$, shocks form at or near the sphere, but shocks are disallowed in the condensate since they represent a violation of the Landau criterion and a breakdown of superfluidity. When $v > v_c$, the condensate evades shocks through a different mode of boundary layer separation. The sphere sheds circular vortex rings that move more slowly than the sphere and form a vortex street that trails behind it, maintained by vortices that the sphere sheds. As the velocity of the ion increases such a shedding becomes more and more irregular. Each ring is born at one particular latitude within the healing layer on the sphere. As it breaks away into the mainstream, it at first contributes a flow that depresses the mainstream velocity on the sphere below critical. As it moves further downstream however, its influence on the surface flow diminishes. The surface flow increases until it again reaches criticality, when a new ring is nucleated and the whole sequence is repeated. The vortex street trailing behind the ion creates a drag on the ion that decreases as the nearest vortex moves downstream, but which is refreshed when a new vortex is born.

Frisch *et al.* (1992) and Winiecki *et al.* (1999) have solved the condensate equation for flow past a circular cylinder, and have confirmed the main features of the scenario just described. In this paper, we present analogous solutions for the more realistic, geometry. By employing a convergent series expansion suitable for $\mathbf{u} = O(c)$, we determine v_c for $\epsilon = 0$ more accurately than before. We confirm this value through numerical integrations at finite ϵ, at the same time obtaining indications of how v_c depends on ϵ. (We should observe here that the criterion $v = c$ for criticality applies only for $\epsilon = 0$. The velocity in a healing layer can exceed c without implying nucleation. For example, $\mathbf{v}$ in a vortex core actually becomes infinite, according to the condensate model!) We also show how and why the vortex ring detaches, not from the equator of the ion, but from a latitude downstream of it. We find how this latitude depends on v.

2 The condensate equation

According to the Bose condensate model, $\psi(\mathbf{x}, t)$ in an assembly of N bosons of mass M, is governed by the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi - \psi\left(E + \frac{1}{2}Mv^2 - V_0|\psi|^2\right), \tag{1}$$

where V_0 is the strength of the $\delta-$function interaction potential between the bosons and E is the single particle energy in the laboratory frame, where the ion moves with velocity v in the positive $z-$direction through fluid at rest at infinity. Equation (1) is written for the ion reference frame, in which the fluid at infinity is moving with velocity v in the negative $z-$direction and the ion is at rest. Thus we require that

$$\psi \to \psi_\infty \exp\left[-\frac{iMvz}{\hbar}\right], \qquad \text{for} \qquad \mathbf{x} \to \infty, \tag{2}$$

where $\psi_\infty = (E/V_0)^{1/2}$.

The ion is modelled as a sphere of radius b that is an infinite potential barrier to the condensate, so that

$$\psi = 0, \qquad \text{at} \qquad r = b. \tag{3}$$

We have here introduced a spherical coordinate system $(r,\ \theta,\ \phi)$, with origin at the centre O of the ion, and with $\theta = 0$ as z-axis. The wavefunction is required to obey the normalization condition on the total number of the bosons $N = \int |\psi|^2\, dV$. The mass density and flux are

$$\rho = M\psi\psi^*, \qquad \mathbf{j} = \frac{\hbar}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{4}$$

Equation (1) can be written in hydrodynamic form through the Madelung transformation,

$$\psi = Re^{iS}, \tag{5}$$

so that

$$\rho = MR^2, \qquad \mathbf{j} = \rho\mathbf{u} = \rho\nabla\phi, \qquad \phi = (\hbar/M)S. \tag{6}$$

The real and imaginary parts of (1) then yield a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{7}$$

and an integrated form of the momentum equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 - \frac{1}{2}v^2 + c^2\left(\frac{\rho}{\rho_\infty} - 1\right) - \frac{\hbar^2}{2M^2}\frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} = 0, \tag{8}$$

the last term of which is often called the "quantum pressure" although it is dimensionally a chemical potential. Also appearing in (8) are the density at infinity ρ_∞ and the speed of sound c:

$$\rho_\infty = M\psi_\infty^2, \qquad c^2 = E/M. \tag{9}$$

We also define the healing length, a, as

$$a = \frac{\hbar}{(2ME)^{1/2}}. \tag{10}$$

The boundary conditions (2) and (3) give

$$\rho \to \rho_\infty, \qquad \mathbf{u} \to -v\mathbf{1}_z, \qquad \text{for} \qquad \mathbf{x} \to \infty, \tag{11}$$

$$\rho = 0, \qquad j_r = 0 \qquad \text{on} \qquad r = b, \tag{12}$$

where $\mathbf{1}_q$ denotes the unit vector in the direction of increasing coordinate q. There is no requirement that $u_r = 0$ on $r = b$; indeed, the problem would be overdetermined if we applied that condition.

3 Asymptotic expansion for velocities up to criticality

In this section, we develop the asymptotic expansion of solutions for small $\epsilon \equiv a/b$. We suppose that the speed of the ion is comparable with the speed of sound c, so that effects of compressibility cannot be ignored. The appropriate non-dimensionalization of (1) is

$$\mathbf{x} \to b\mathbf{x}, \qquad t \to \left(abM/\hbar\right)t, \qquad v \to \left(\hbar/aM\right)U, \qquad \psi \to \psi_\infty\psi. \tag{13}$$

Subcritical flow is steady in the ion reference frame, and the Madelung equations are therefore

$$\epsilon^2\nabla^2 R - R(\nabla S)^2 = (R^2 - 1 - U^2)R, \tag{14}$$

$$R\nabla^2 S + 2\nabla R \cdot \nabla S = 0. \tag{15}$$

The quantum pressure term, $\epsilon^2\nabla^2 R$ is negligibly small in the far field but is of major importance in the boundary layer, for which we set $r = 1 + \epsilon\xi$, and expand R and S as

$$R(\xi,\theta) \quad = \widehat{R_0}(\xi,\theta) + \epsilon\widehat{R_1}(\xi,\theta) + \epsilon^2\widehat{R_2}(\xi,\theta) + \cdots, \tag{16}$$

$$S(\xi,\theta) \quad = \widehat{S_0}(\xi,\theta) + \epsilon\widehat{S_1}(\xi,\theta) + \epsilon^2\widehat{S_2}(\xi,\theta) + \cdots. \tag{17}$$

Equations (16) and (17) give $\partial\widehat{S_0}/\partial\xi = \partial\widehat{S_1}/\partial\xi = 0$, so that

$$\widehat{S_0} = \widehat{S_0}(\theta), \qquad \widehat{S_1} = \widehat{S_1}(\theta) \tag{18}$$

where $\widehat{S_0}(\theta)$ and $\widehat{S_1}(\theta)$ are to be determined by matching to u_θ on $r = 1$ in the mainstream. After we substitute the $\widehat{S_0}$ into (14), it becomes to leading order

$$\frac{\partial^2\widehat{R_0}}{\partial\xi^2} - \widehat{R_0}^3 + \widehat{R_0}\left[1 + U^2 - (\widehat{S_0}'(\theta))^2\right] = 0, \tag{19}$$

so that

$$\widehat{R_0} = g(\theta)\tanh(g(\theta)\xi/\sqrt{2}), \tag{20}$$

where

$$g(\theta) = \sqrt{[1 + U^2 - (\widehat{S_0}'(\theta))^2]}. \tag{21}$$

The equation governing $\widehat{S}_2$ is

$$\frac{\partial}{\partial \xi}\left(\widehat{R_0}^2 \frac{\partial \widehat{S}_2}{\partial \xi}\right) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\widehat{R_0}^2 \sin\theta \frac{\mathrm{d}\widehat{S}_0}{\mathrm{d}\theta}\right]. \tag{22}$$

This gives $\widehat{S}_2$ as

$$\widehat{S}_2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(h(\xi,\theta)\sin\theta\frac{\mathrm{d}\widehat{S}_0}{\mathrm{d}\theta}\right) + \zeta_2(\theta), \tag{23}$$

where

$$h(\xi,\theta) = \int_0^{\xi} \frac{d\xi'}{\widehat{R_0}^2(\xi',\theta)} \int_0^{\xi'} \widehat{R_0}^2(\xi'',\theta)d\xi'' = \frac{1}{2}\xi^2 - \frac{\sqrt{2}\xi}{g(\theta)}\coth\left(\frac{g(\theta)\xi}{\sqrt{2}}\right), \tag{24}$$

and $\zeta_2(\theta)$ is a function of integration that can be determined by matching to the mainstream.

To leading order, the mainstream flow is the classical inviscid compressible flow past a sphere, and is governed by

$$R^2 = 1 + U^2 - (\nabla S)^2, \tag{25}$$

$$R^2\nabla^2 S + \nabla R^2 \cdot \nabla S = 0. \tag{26}$$

We substitute the first equation of this system into the second to obtain an equation for S alone. We then expand S as in (17) and then expand S_0, S_1, $\cdots$ in powers of U, e.g.

$$S_0 = US_{11}(r)P_1(\cos\theta) + U^3(S_{31}(r)P_1(\cos\theta) + S_{33}(r)P_3(\cos\theta)) + \cdots. \tag{27}$$

For U close to the speed of sound, $c = 1/\sqrt{2}$, we do not expect (27) to converge fast enough to be useful, but the critical U which we are trying to determine is approximately $2c/3$ and for such values of U the expansion (27) converges fast enough. We expand S to the U^{11} term in order to get an estimate for the critical velocity of nucleation, U_c, accurate to 0.1%. The first few equations for the mainstream are

$$\frac{d^2S_{11}}{dr^2}+\frac{2}{r}\frac{\partial S_{11}}{r}-\frac{2S_{11}}{r^2}=0, \tag{28}$$

$$\begin{aligned}\frac{d^2S_{31}}{dr^2}+\frac{2}{r}\frac{dS_{31}}{dr}-\frac{2S_{31}}{r^2}=&\frac{9}{5}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2S_{11}}{\mathrm{d}r^2}+\frac{2S_{11}^2}{r^2}\frac{\mathrm{d}^2S_{11}}{\mathrm{d}r^2}\\&+\frac{6}{5r}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^3+\frac{2S_{11}}{5r^2}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^2-\frac{8S_{11}^3}{5r^4},\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{d^2S_{33}}{dr^2}+\frac{2}{r}\frac{dS_{33}}{dr}-\frac{12S_{33}}{r^2}=&\frac{6}{5}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^2\frac{\mathrm{d}^2S_{11}}{\mathrm{d}r^2}-\frac{2S_1^2}{5r^2}\frac{\mathrm{d}^2S_{11}}{\mathrm{d}r^2}\\&+\frac{4}{5r}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^3-\frac{12S_1}{5r^2}\left(\frac{\mathrm{d}S_{11}}{\mathrm{d}r}\right)^2+\frac{8S_{11}^3}{5r^4}.\end{aligned} \tag{30}$$

Solving these, we obtain

$$S_{11}=-\left(r+\frac{1}{2r^2}\right),\quad S_{33}=-\frac{3}{88r^8}+\frac{3}{5r^5}+\frac{3}{5r^2}+\frac{C_1}{r^4},\quad S_{31}=-\frac{1}{12r^8}+\frac{2}{5r^5}+\frac{C_2}{r^2}. \tag{31}$$

To carry out the asymptotic matching, we substitute $r=1+\epsilon\xi$ in the expressions for $S_{11}, S_{31}, S_{33}, \cdots$, expand the solution in powers of ϵ, and match it to the boundary layer solution. The first few terms of the resulting mainstream solution are

$$
\begin{aligned}
S_0(r,\theta) = -\frac{3U\cos\theta}{2r^2} + U^3\Bigg(&\left[-\frac{1}{12r^8}+\frac{2}{5r^5}-\frac{2}{3r^2}\right]\cos\theta \\
&+\left[-\frac{3}{88r^8}+\frac{3}{5r^5}-\frac{54}{55r^4}+\frac{3}{5r^2}\right]P_3(\cos\theta)\Bigg) \\
+U^5\Bigg(\Bigg[&-\frac{1901}{18480r^{14}}+\frac{1739}{2310r^{11}}-\frac{5589}{21175r^{10}} \\
&-\frac{1766}{1155r^8}+\frac{972}{1925r^7}+\frac{776}{525r^5}-\frac{1470911}{1016400r^2}\Bigg]\cos\theta \\
+\Bigg[&-\frac{179}{2244r^{14}}+\frac{7256}{8085r^{11}}-\frac{2274}{3575r^{10}}-\frac{523}{220r^8} \\
&+\frac{624}{275r^7}+\frac{152}{75r^5}-\frac{72781521}{23823800r^4}+\frac{106}{75r^2}\Bigg]P_3(\cos\theta) \\
+\Bigg[&-\frac{1945}{140448r^{14}}+\frac{2291}{9240r^{11}}-\frac{87}{385r^{10}}-\frac{295}{182r^8} \\
&+\frac{240}{77r^7}-\frac{51393}{434720r^6}-\frac{68}{21r^5}+\frac{24}{11r^4}-\frac{10}{21r^2}\Bigg]P_5(\cos\theta)\Bigg)+\cdots.
\end{aligned}
\tag{32}
$$

From this we can determine the first function of (18):

$$
\begin{aligned}
S_0(\theta) = &-\frac{3}{2}U\cos\theta - \frac{7}{20}U^3\cos\theta + \frac{81}{440}U^3P_3(\cos\theta) \\
&+U^5\left[-\frac{14693}{24200}\cos\theta + \frac{1560249}{3403400}P_3(\cos\theta) - \frac{31857}{217360}P_5(\cos\theta)\right]+\cdots.
\end{aligned}
\tag{33}
$$

The maximum flow velocity is attained on the equator and is

$$
\begin{aligned}
u_\theta(1,\tfrac{1}{2}\pi) = 3U/2 + 0.626136U^3 + 1.56961U^5& \\
+5.18161U^7 + 19.9015U^9 + 26.8951U^{11} + \cdots&,
\end{aligned}
\tag{34}
$$

where we have here included terms up to order U^{11}. The flow (34) reaches the speed of sound if the far field velocity U is approximately 0.415. An idea of the accuracy of this value of U_c is obtained by comparing it with the final term of (34), which for $U_c = 0.415$ is 0.0017. The result also agrees very well with the numerical calculations of §5 for small ϵ. By comparing 0.415 with $2c/3 \approx 0.471$, we gain an impression of the importance of compressibility in determining U_c in the condensate model.

The $\mathcal{O}(\epsilon)$ contribution to the mainstream solution satisfies the following equations

$$R_0 R_1 = -\nabla S_0 \cdot \nabla S_1, \tag{35}$$

$$2R_0 R_1 \nabla^2 S_0 + R_0^2 \nabla^2 S_1 + 2\nabla\Big(R_0 R_1\Big) \cdot \nabla S_0 + \nabla R_0^2 \cdot \nabla S_1 = 0. \tag{36}$$

We substitute the first equation of this system into the second to obtain the equation on S_1 along and expand S_1 as in (27) to the U^5 term.The first few terms of the resulting mainstream solution are

$$S_1(r,\theta) = U\frac{C_1}{r^2}\cos\theta + U^3\Bigg(\left[\frac{C_1}{2r^8} - \frac{8C_1}{5r^5} + \frac{C_2}{r^2}\right]\cos\theta + \left[\frac{9C_1}{44r^8} - \frac{12C_1}{5r^5} - \frac{6C_1}{5r^2} + \frac{C_3}{r^4}\right]P_3(\cos\theta)\Bigg) + \cdots. \tag{37}$$

The unknown constants are found by matching the boundary layer solution (23) to (37). We substitute $r = 1 + \epsilon\xi$ in (37), expand the solution in powers of ϵ, and notice that the last term in (24) is $\mathcal{O}(\xi)$ for $\xi \to \infty$. We expand the corresponding term of S_2 for large ξ in powers of U and in the Legendre polynomials, and set the coefficients of the resulting expansion equal to the corresponding coefficients at the order ϵ in the expansion of (37). This defines the constants in (37) as

$$C_1 = -\frac{3}{\sqrt{2}}, \qquad C_2 = -\frac{13\sqrt{2}}{5}, \qquad C_3 = -\frac{1863}{220\sqrt{2}},$$
$$C_4 = -\frac{4414007}{338800\sqrt{2}}, \qquad C_5 = -\frac{1352579157}{47647600\sqrt{2}}, \qquad C_6 = \frac{24662449}{760760\sqrt{2}}. \tag{38}$$

On the equator of the sphere the ϵ term of the expansion for the velocity becomes

$$\tilde{u_\theta} = \frac{3}{\sqrt{2}}U + \frac{197}{44\sqrt{2}}U^3 + \frac{186422556609}{11380969600\sqrt{2}}U^5 + \cdots. \tag{39}$$

The interesting question is whether the S_1 term increases or decreases U_c. This raises the philosophical point touched on in §1: is the Landau criterion precise when we go beyond the leading term in the $\mathbf{u}$ expansion if the mainstream by including

ϵ term associated with the quantum pressure? We are encouraged to believe it is; because of the upward curvature of the dispersion curve associated with the condensate model, the speed of long wavelength sound plausibly sets the stability limit for all disturbances. We therefore now set $u_\theta(1\frac{1}{2}\pi) = c$, where u_θ includes (34) and (39). We find that the S_1 decreases U_c. For example, for $\epsilon = 0.1$ the flow reaches the velocity of sound if the far field velocity U is approximately 0.37, which agrees very well with our numerical calculations of §5.

4 Asymptotic expansion for the cylinder

In considering the shedding of line vortices from a moving cylinder, Frisch *et al.* (1992) gave an argument for the critical velocity that we shall now test through an expansion of the same type as that of the preceding section. Instead of (27), we now use a Fourier expansion in θ, which is one of the cylindrical coordinates (r, θ, z), with $\theta = 0$ along $\mathbf{U}$:

$$S_0 = US_{11}(r)\cos\theta + U^3(S_{31}(r)\cos\theta + S_{33}(r)\cos 3\theta) + \cdots. \tag{40}$$

The mainstream solution is found to be

$$\begin{aligned} S_0(r,\theta) = &- U\left(r + \frac{1}{r}\right)\cos\theta \\ &+ U^3\left[\left(-\frac{1}{6r^2} + \frac{1}{r^3} - \frac{13}{6r}\right)\cos\theta + \left(-\frac{1}{r^3} + \frac{1}{2r}\right)\cos 3\theta\right] \\ &+ U^5\left[\left(-\frac{7}{30r^9} + \frac{3}{2r^7} - \frac{43}{12r^5} + \frac{35}{6r^3} - \frac{479}{60r}\right)\cos\theta\right. \\ &\quad + \left(-\frac{1}{36r^9} + \frac{4}{15r^7} - \frac{3}{2r^5} + \frac{43}{30r^3} + \frac{19}{12r}\right)\cos 3\theta \\ &\quad \left. + \left(\frac{7}{20r^5} - \frac{1}{2r^3} - \frac{1}{4r}\right)\cos 5\theta\right] + \cdots \end{aligned} \tag{41}$$

The boundary layer function corresponding to (18) is

$$\widehat{S}_0(\theta) = -\,2U\cos\theta + U^3\left(-\frac{4}{3}\cos\theta + \frac{1}{3}\cos 3\theta\right)$$
$$+\,U^5\left(-\frac{67}{15}\cos\theta + \frac{79}{45}\cos 3\theta - \frac{2}{5}\cos 5\theta\right)$$
$$+\,U^7\left[-\frac{251}{12}\cos\theta + \frac{203933}{18900}\cos 3\theta - \frac{80113}{18900}\cos 5\theta + \frac{11}{14}\cos 7\theta\right] + \mathrm{O}(U^9). \tag{42}$$

The maximum flow velocity, which occurs on the cylinder equator, is

$$u_\theta(1, \tfrac{1}{2}\pi) = 2U + 7U^3/3 + 176U^5/15 + 79.9809U^7 + 552.181U^9 + 4471.18U^{11} + \cdots \tag{43}$$

This reaches the velocity of sound for $U = U_c \approx 0.30$. (To illustrate the convergence of (43), we note that the U^{11} term is only 0.0066 for $U = 0.30$.)

According to Frisch *et al.* (1992), criticality is reached when, in our nondimensional units, the velocity exceeds $\rho/2$ anywhere in the mainstream. In the present problem, this predicts that the critical velocity is attained first at the equator of the cylinder, and (43) gives $U_c \approx 0.26$, which is significantly less than our value, and is also less than the value $U_c \approx (0.45 \pm 0.01)c = 0.318 \pm 0.007$ for a finite ϵ, obtained by Winiecki *et al.* (1999) from their numerical integrations.

5 Numerical calculations

In this section we present some results from a numerical calculations for the axisymmetric flow around the sphere and the nucleation of vortex rings from it. We used a third non-dimensionalization of (1):

$$\mathbf{x} \to a\mathbf{x}, \qquad t \to (a^2M/\hbar)t, \qquad v \to (\hbar/aM)U, \qquad \psi \to \left(\psi_\infty \mathrm{e}^{-iUz}\right)\psi, \tag{44}$$

the last of which removes a uniform flow $-v\mathbf{1}_z$ everywhere, so that

$$\psi \to 1 \qquad \text{as} \qquad r \to \infty. \tag{45}$$

Equation (1) becomes

$$-2\mathrm{i}\frac{\partial\psi}{\partial t}+2\mathrm{i}U\frac{\partial\psi}{\partial z}=\nabla^2\psi+\psi(1-|\psi|^2), \tag{46}$$

the solution to which must satisfy (45) and

$$\psi=0 \qquad \text{on} \qquad r=b/a. \tag{47}$$

We employed a finite difference scheme to solve (46) in the axisymmetric case in which ψ depends only on r and θ, and in which therefore

$$\begin{aligned}-2\mathrm{i}\frac{\partial\psi}{\partial t}+2\mathrm{i}U\cos\theta\psi_r-2\mathrm{i}U\frac{\sin\theta}{r}\psi_\theta=&\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}\\&+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\psi(1-|\psi|^2).\end{aligned} \tag{48}$$

The integration box was chosen as $[b/a, r_1]\times[0,\pi]$. One of the main considerations in choosing the integration scheme was that outgoing sound waves should escape from the integration box. We used the Raymond-Kuo (1984) radiation boundary condition on $r=r_1$. In time stepping the leap-frog scheme was implemented with a backward Euler step every 100 steps to prevent the even-odd instability. In space we used a 4th order finite difference scheme together with a 2nd order scheme close to the boundary $r=r_1$. The code was tested against the asymptotic solutions of §3. The initial condition for velocities slightly larger than U_c was chosen as $\psi=\tanh(\xi/\sqrt{2})$. The numerical scheme does not conserve energy but the dissipation of energy is very small. When the reflective boundary conditions were used instead of the radiative ones, the energy loss due to the dissipative character of the scheme did not exceed $10^{-4}\%$ per 1000 time steps.

Our numerical work strongly suggests that the value of U_c (≈ 0.415) obtained in §3 for $\epsilon=0$ is correct. We also found that U_c decreases slowly with increasing ϵ, in agreement with the results of §3.

Figure 1 shows the formation of the ring from a sphere of radius 10 moving with the supercritical velocity $U=0.42$. After the ion emits a vortex ring, the flow associated with the ring at first makes the total fluid velocity subcritical everywhere.

The self-induced velocity of the ring is less than the velocity of the ion, so that the ring gradually falls astern of the ion and the total fluid velocity builds up until it again reaches criticality on the surface of the ion. The vortex ring emission follows the same scenario as that observed by Frisch *et al.* (1992) for vortex pair nucleation from a cylinder; see §1. What came as a surprise is that, although the maximum velocity of the compressible flow is always attained on the equator of the sphere, the vortex ring nucleates from the sphere downstream of the equator, at $\theta_c > \frac{1}{2}\pi$. This is clearly seen in Figure 1(a), which shows the birth of the first vortex after the motion of the ion has been initiated. The first nucleation seems to be the result of an instability of the critical flow (obtained from §3, as described above). The second nucleation is influenced by the presence of the first ring; see Figure 1(b). It is therefore takes place at a different location on the ion surface, as is readily seen in Figure 1(c).

It is difficult to extend the theory of §3 to cover the time-dependent supercritical state that arises when vortices are nucleated. It is however comparatively easy to generalize (18)-(21) for the healing layer. We find that

$$\widehat{S}_0 = \widehat{S}_0(\theta, t), \tag{49}$$

where

$$\frac{\partial^2 \widehat{R_0}}{\partial \xi^2} - \widehat{R_0}^3 + \widehat{R_0}\Big[1 + U^2 - \Big(\frac{\partial \widehat{S}_0}{\partial \xi}\Big)^2 - 2\frac{\partial \widehat{S}_0}{\partial t}\Big] = 0. \tag{50}$$

The relevant solution is

$$\widehat{R_0} = g(\theta, t)\tanh(g(\theta, t)\xi/\sqrt{2}), \tag{51}$$

where now

$$g(\theta, t) = \sqrt{\Big[1 + U^2 - \Big(\frac{\partial \widehat{S}_0}{\partial \xi}\Big)^2 - 2\frac{\partial \widehat{S}_0}{\partial t}\Big]}. \tag{52}$$

The $\partial\widehat{S}_0/\partial t$ term in (52) is highly significant: the nucleation of a vortex ring at latitude θ_c occurs at time t_c, when

$$g(\theta_c, t_c) = 0, \tag{53}$$

with $g > 0$ for all other θ in $0 \leq \theta \leq \pi$. Thus, the nucleation of a vortex ring represents a breakdown of the healing layer. This may be traced to the growth in importance of the first term in (8) at θ_c and the concomitant decrease in significance of the final, quantum pressure. This decrease implies that, at nucleation, $\partial R/\partial \xi$ at θ_c is no larger than ∇R in the mainstream, i.e., the mainstream and healing layer have become temporarily connected. This provides the channel through which the vortex escapes from the ion. It may be noted that the breakdown ($g = 0$) of the healing layer in supercritical ion is not directly linked to the criterion ($|\mathbf{v}| = c$) used to determine the critical state for the steady subcritical solutions for $\epsilon = 0$. This explains why $\theta_c > \frac{1}{2}\pi$, even though the maximum u_θ on the sphere still occur on the equatorial plane ($\theta = \pi/2$).

As it is impractical to expand the theory of §3 for the mainstream to the time-dependent case, and then to determine $\widehat{S}_0$ and g by matching to the healing layer, we must determine $\widehat{S}_0$ and g from the numerical results for small ϵ. Figures 2 show how g evolves for $\epsilon = 0.1$ and the velocity of the far field $U = 0.42$. The first vortex begins to be formed when g becomes zero at $\theta_c \approx 120°$ at $t_c \approx 35$; see Figure 2(a), where a second minimum in g can also be seen on the $t = 35$ curve near $\theta_c \approx 110°$. This second minimum develops, and becomes zero when the next vortex is nucleated; see Figure 2(b). The breakdown of the healing layer is very evident in Figures 2, and is responsible for the two blips seen on the sphere in Figure 1(a) and, at different θ−locations, in Figure 1(c). It is also clear that, as the healing layer thickens at θ_c, it provides the core of the nascent vortex. The angle θ_c at which the first vortex is nucleated, changes with U, as shown in Table 1.

Table 1

U	0.39	0.40	0.42	0.45
θ_c	100°	111°	120°	126°

6 Conclusions

We have used the Bose condensate model of superfluid helium to clarify the process through which a moving ion generates vortices if its velocity, v, exceeds a certain critical value, v_c, of the same order as the Landau critical velocity (which for the condensate, which has no roton minimum on its dispersion curve, is the speed of sound, c). To some extent, v_c depends on ϵ, the ratio of the healing length to the sphere radius. We have determined v_c analytically to leading order in ϵ by equating to c the flow velocity at the equator of the ion. This does not mean, however, that the vortex ring is emitted from the equator when $v > v_c$, as was supposed by Strayer *et al.* (1971), Muirhead *et al.* (1984), and Frisch *et al.* (1992). This was demonstrated through direct numerical simulations for small but finite ϵ. We have shown, through asymptotic analysis, that the vortex rings emerge from singularities that develop periodically in the healing layer at some particular latitudes θ_c. We have found that θ_c increases with v, i.e., the point of detachment moves towards the rear stagnation point ($\theta = \pi$).

The development of the singularity is intimately linked to the time-dependence of the mainstream supercritical flow, which fluctuates as the vortices move downstream to join the train of rings following the ion. We have not analyzed what happened the first time that vortex is created, but we surmise that in this case the singularity develops as the result of the instability of the mainstream flow. Thereafter, time-dependence is assured through the ring (and later rings) trailing behind the ion.

The breakdown of the healing layer is the analogue for the superfluid of boundary layer separation in high Reynolds number viscous flow, and this explains the choice of subtitle for our paper.

Acknowledgment

This work was supported by the NSF grant DMS-9803480.

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Figure 1: The density plot of cross section of the solution of (48) for the flow around a sphere of radius 10 moving to the right with velocity 0.42 at (a) t = 68, (b) t = 134, and (c) t = 223. Vortex rings appear as white circles close to the sphere and gradually fall astern of the ion.

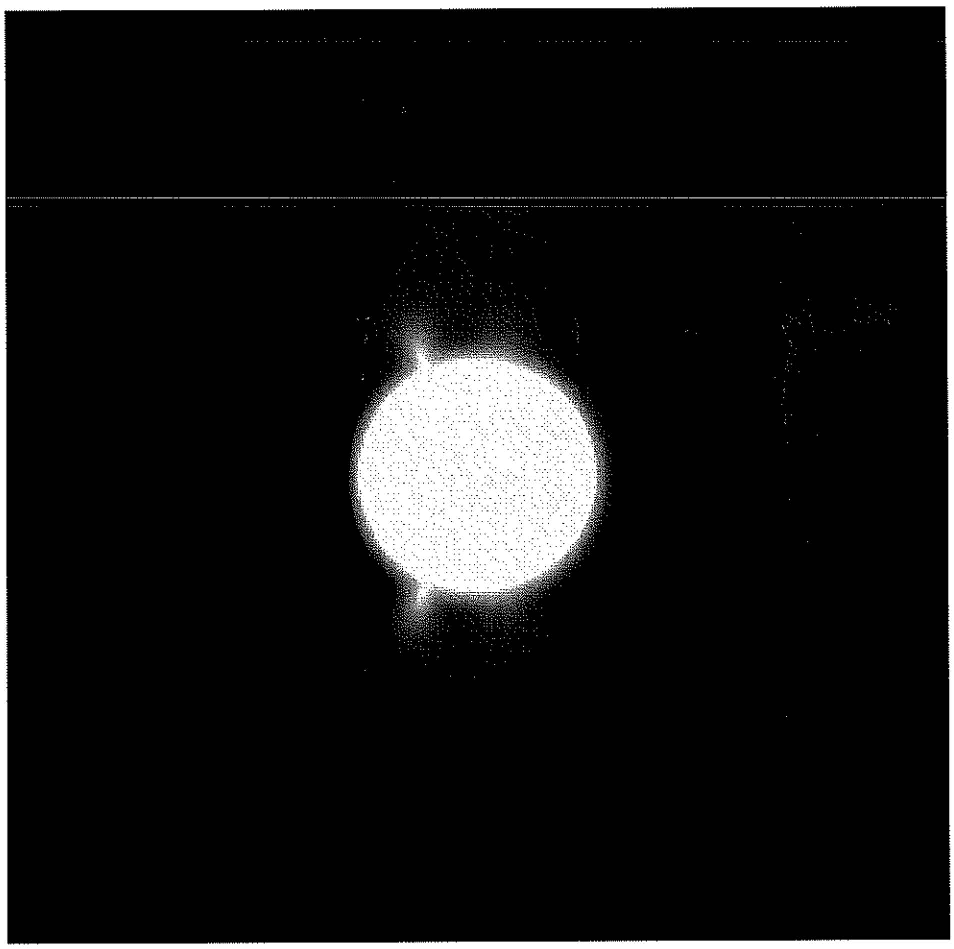

Figure 1(b)

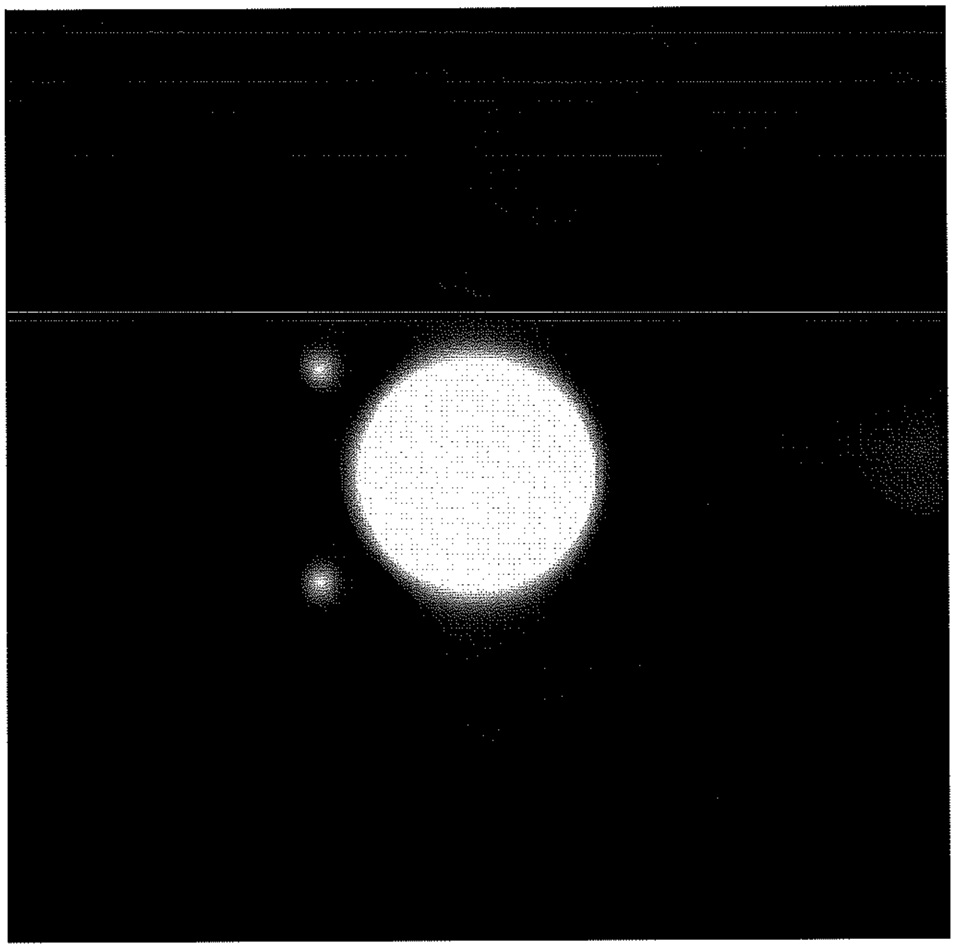

Figure 1(c)

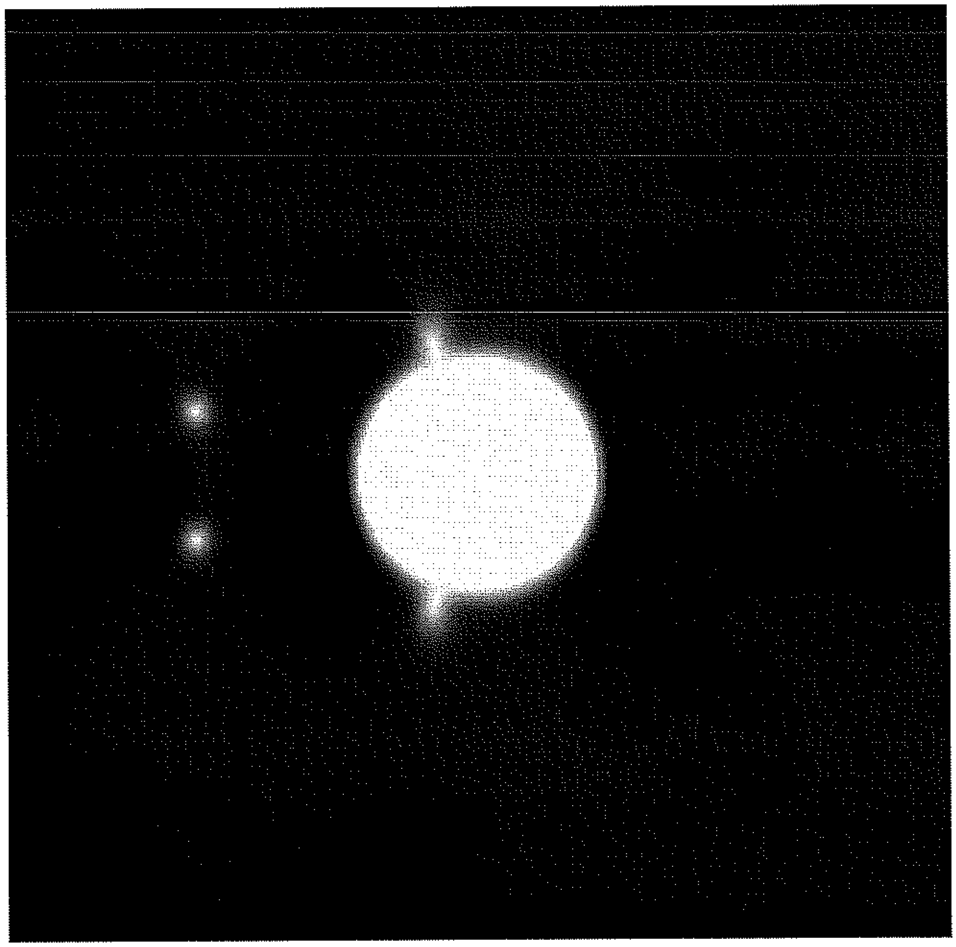

Figure 2: Time evolution of $g(\theta)^2$ defined in (52) for the flow around a sphere of radius 10 immediately before the nucleation of the first (a) and the second (b) vortex rings.