QUALIFYING EXAM

Geometry/Topology

March 2023

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let M, N be smooth manifolds and $F : M \to N$ a smooth proper map.
   (a) Show that F maps closed sets to closed sets.
   (b) Show that the set of regular values is open.
   (c) Let $C \subset N$ be compact. Show that for every open set $U \subset M$ containing $F^{-1}(C)$ there is an open set $V \subset N$ containing C, such that $F^{-1}(V) \subset U$.

2. Consider a smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$.
   (a) When n is even show that F has a fixed point.
   (b) When n is odd give an example where F does not have a fixed point.

3. Let
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
be a 2-form defined on $\mathbb{R}^3 - \{0\}$ and $S^2 \subset \mathbb{R}^3$ the unit sphere.
   (a) Compute $\int_{S^2} i^*\omega$, where $i : S \to \mathbb{R}^3$ is the inclusion.
   (b) Compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ is defined by $j(x, y, z) = (2x, 3y, 5z)$.

4. Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

5. Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.
   (a) Show that $\mathbb{R}^n \backslash M^m$ is connected when $m \leq n - 2$.
   (b) When $m = n - 1$ show that $\mathbb{R}^n \backslash M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

6.
   (a) If X is a finite CW complex and $\tilde{X} \to X$ is a path-connected n-fold covering map, then show that the Euler characteristics are related by the formula
$$\chi(\tilde{X}) = n\chi(X).$$
   (b) Let $X = \Sigma_g$ be a closed genus g surface. What path-connected, closed surfaces can cover X?

7. A group G is divisible if for all n, the map $g \mapsto g^n$ from G to itself is surjective. Show that if X is a path-connected CW-complex and if $\pi_1(X, x)$ is a divisible group, then the only path connected finite cover of X is X itself. (Hint: This can be proven directly or by first showing that a divisible group has no finite index subgroups.)

8. Let M^n be an n-manifold, and consider a small disk D^n embedded in M^n. Show that the inclusion
$$\overline{M^n - D^n} \hookrightarrow M^n$$
induces an isomorphism on π_1 if $n \geq 3$ and a surjection if $n \geq 2$.

9. Find, as a function of n and m, the homology groups
$$H_*(\mathbb{RP}^{n+m}, \mathbb{RP}^n; \mathbb{Z}).$$

10. Consider the CW-complexes $A = S^n \vee S^n$, $X = S^n \times S^n$, and $B = S^n \times [0,1]/ * \times [0,1]$, where $*$ is the basepoint of S^n. There are inclusions $A \hookrightarrow X$ given by the pairs of points where at least one is the basepoint and $A \hookrightarrow B$ which takes one S^n to $S^n \times 0$ and the other to $S^n \times 1$. Compute the homology of
$$Y = X \cup_A B.$$