QUALIFYING EXAM
Geometry/Topology
September 2023

Attempt all ten problems. Each problem is worth 10 points. You must fully justify your answers.

1. Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

2. Let ω be a closed 2-form on a smooth manifold M and let X, Y be smooth vector fields on M. Show that if $i_X\omega = i_Y\omega = 0$, then $i_{[X,Y]}\omega = 0$.

3. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.

   (a) (3 pts) Show that d_f is a cochain map, i.e., $d_f \circ d_f = 0$.

   (b) (7 pts) Let $H^i_f(M)$ be the ith cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^0_f(M) \simeq \mathbb{R}$ when M is the real line $\mathbb{R}$.

4. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost all $a \in \mathbb{R}^n$, the translate $X + a := \{x + a \mid x \in X\}$ intersects Y transversely.

5. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and let C be the curve which is the image of the line $\{2x - 5y = 0\} \subset \mathbb{R}^2$ under the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

   (a) Write a differential form on T^2 which represents the Poincaré dual to C.

   (b) Is there a differential form which represents the Poincaré dual to C and is zero on a neighborhood of the point $(0,0) \in T^2$?

6. Compute the integral homology groups of the complex projective space $\mathbb{CP}^n$. If n is even, prove that it does not cover any manifold except itself.

7. Let $X = \Sigma_g$ and $Y = \Sigma_h$ be surfaces of genus g and h respectively, with $0 < g < h$. Prove that every map $X \to Y$ induces the zero map on the second homology H_2. Construct a map $X \to Y$ which induces a non-zero map on the first homology H_1.

8. Consider the following group with $2n$ generators and 1 relation

$$G_n = \langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle.$$

   For which pairs (m, n) does G_n contain a finite index subgroup isomorphic to G_m?

9. Define the orientation double cover of a manifold. Explicitly identify the space which is the orientation double cover of the real projective plane $\mathbb{RP}^n$. (Hint: $\mathbb{RP}^n$ is the quotient of S^n by the antipodal map; is the antipodal map orientation-preserving or orientation-reversing?)

10. Let D^2 be the unit disk in $\mathbb{C}$, and let $S^1 = \partial D^2$. Let $X = D^2 \times S^1 \times \{0, 1\} / \sim$ where

$$(x, y, 0) \sim (xy^5, y, 1)$$

for all $x, y \in S^1$. Compute the homology groups of X.