Qualifying Exam, Spring 2023

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Consider the problem of fitting a function of the form

$$f(x) = a + b\cos(x) + c\cos(2x)$$

to the data

x	0	$\pi/2$	π	$3\pi/2$
y	1	1	1	0

(a) Give a derivation of the set of linear equations whose solution determines the coefficients a, b, c so that $\sum_{i=1}^{4} |f(x_i) - y_i|^2$ is a minimum.

(b) Solve the linear system and give the coefficients a, b, c you obtain.

[2] (5 Pts.) (a) Derive the coefficients of a 4th order finite difference approximation to $\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$ that utilizes function values spaced h apart.

(b) Justify that the coefficients you determined in (a) result in a 4th order approximation.

[3] (5 Pts.) The Midpoint rule for numerical integration, with error term, is given below,

$$\int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi),$$

where $f \in C^2[x_{-1}, x_1]$, $x_{-1} < \xi < x_1$, and $x_0 - x_{-1} = x_1 - x_0 = h > 0$.

Determine the values of n and h required to approximate $\int_0^2 e^{2x} \sin 3x \, dx$ using the Composite Midpoint rule to within 10^{-4} (justify your answer).

[4] **(5 Pts.)** **(a)** Give the Bisection Method and show its convergence.

(b) Determine the number of iterations necessary to solve $x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3}, using the Bisection method with $a_1 = 1$ and $b_1 = 2$ (justify your answer).

[5] **(10 Pts.)** Let $\alpha < 0$ and consider the numerical method

$$y^{n+1} = e^{\alpha h} y^n + \frac{\left(e^{\alpha h} - 1\right)}{\alpha} f(y^n)$$

to determine approximate solutions of the initial value problem

$$\frac{dy}{dt} = \alpha\, y + f(y)$$

$y(0) = y_0$ for $t \in [0, \mathrm{T}]$, with $f(y)$ smooth and having global Lipschitz constant K.

(a) Let $h = \dfrac{\mathrm{T}}{\mathrm{N}}$ be the timestep size, derive an expression for the leading term of the local truncation error for the method.

(b) Let $\tilde{e}_{\mathrm{N}} = y(\mathrm{T}) - y_{\mathrm{N}}$ designate the error in the solution at time T obtained using N timesteps with stepsize $h = \dfrac{\mathrm{T}}{\mathrm{N}}$. Give a proof that $\tilde{e}_{\mathrm{N}} \to 0$ as $\mathrm{N} \to \infty$ (or equivalently $h \to 0$).

[6] **(10 Pts.)** Consider the second order partial differential equation

$$u_{tt} - 2\,a\,u_{xt} + (a^2 + c^2)\,u_{xx} = 0$$

with a and c positive constants. This is to be solved for $t > 0$, $0 \le x \le 1$, with periodic boundary conditions in x and smooth periodic initial data $u(x, 0) = v(x)$, $u_t(x, 0) = w(x)$.

(a) Show that this is a well posed problem.

(b) Construct a second order accurate and convergent finite difference approximation.

Justify your answers

[7] (10 Pts.) Consider the differential equation

$$u_t + (u^2)_x = c\,u_{xx}$$

with $c > 0$. This is to be solved for $t > 0$, $0 \leq x \leq 1$, with initial data $u(x,0) = v(x)$, with periodic boundary conditions in x.

Devise a convergent finite difference equation which has a maximum and minimum principle, i.e. the maximum of the discrete solution is not increasing in time and the minimum of the discrete solution is not decreasing in time. Show that this remains true even as $c \to 0$.

Justify your answers

[8] (10 Pts.) Consider the *biharmonic problem* in a two-dimensional domain Ω with sufficiently smooth boundary,

$$\triangle\triangle u = f \text{ in } \Omega,$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma = \partial\Omega,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction to the boundary Γ.

(a) Formally show using a Green's formula that, for any $u \in H^2(\Omega)$ satisfying the above boundary conditions, we have

$$\int_\Omega |\triangle u|^2 dxdy = \int_\Omega \left\{(u_{xx})^2 + (u_{yy})^2 + (u_{xy})^2 + (u_{yx})^2\right\} dxdy.$$

(b) Derive a weak variational formulation of the biharmonic problem and show that this has a unique solution u in an appropriate space of functions that you will specify. Assume that $f \in L^2(\Omega)$. Justify your answers.

(c) Briefly describe a finite element approximation of the problem using P_5 elements and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution.

(d) Assume convexity and sufficient regularity of the domain Ω. State a standard error estimate for the approximation.