Qualifying Exam, Fall 2023

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Consider the problem of fitting a function of the form

$$p(x) = a + b\,(2x - 1)$$

to the first three terms of a Taylor expansion of e^x,

$$f(x) = 1 + x + \frac{1}{2}x^2$$

over an interval $[0, 1]$.

(a) Give a derivation of the set of linear equations whose solution determines the coefficients a, b so that $\int_0^1 |p(x) - f(x)|^2\, dx$ is a minimum.

(b) Solve the linear system and give the coefficients a, b you obtain.

[2] (5 Pts.) Consider Newton's method for determining the approximate roots of a function, $f(x) = x\, e^{-x}$.

Determine a value, $\bar{x}$, so that for any initial iterate $x^0 > \bar{x}$ the subsequent Newton iterates will diverge, e.g. $\lim_{n\to\infty} |x^n| = \infty$. Justify your answers.

[3] (5 Pts.) An error expansion for the forward-difference approximation to the derivative can be expressed as

$$f'(x_0) = \frac{1}{h}\Big[f(x_0 + h) - f(x_0)\Big] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

Use Richardson extrapolation to derive the coefficients of an $O(h^3)$ difference approximation for $f'(x_0)$.

[4] **(5 Pts.)** Derive the coefficients of a Gaussian quadrature formula of the form $\sum_{i=1}^{n} c_i f(x_i)$ with $n = 2$ to approximate the integral $\int_{-1}^{1} f(x)dx$. What is the maximal degree of the polynomial for which this approximation is exact?

[5] **(10 Pts.)** Consider the initial value problem

$$\frac{d^2y}{dt^2} + \alpha\, y = 0 \quad y(0) = a \text{ and } \frac{dy}{dt}(0) = b \quad \text{(IVP)}$$

for $t \in [0, 1]$ and α is a postive constant.

(a) Give the initial value problem for a two dimensional first order system that is equivalent to (IVP).

For a two dimensional system of ODE's of the form

$$\begin{aligned} \frac{dp}{dt} &= q \\ \frac{dq}{dt} &= f(p) \end{aligned}$$

consider the numerical method

$$\begin{aligned} p^{n+\frac{1}{2}} &= p^{n-\frac{1}{2}} + h\, q^{n} \\ q^{n} &= q^{n-1} + h\, f(p^{n+\frac{1}{2}}) \end{aligned}$$

with a uniform timestep $h = \frac{1}{\mathrm{N}}$ and $p^{\frac{1}{2}}$ obtained using $p^{\frac{1}{2}} = p^{0} + \dfrac{h}{2} q^{0}$

(b) Derive the order of the local truncation error for this method. Show your work.

(c) For this method applied to the system of equations in **(a)**, derive the relation between the error in the approximate values at step n, $\tilde{e}^{\,n}$, and the error at step $n-1$, $\tilde{e}^{\,n-1}$, where the vector of approximate values, $\vec{v}^{\,n}$, and the error in the approximate values, $\tilde{e}^{\,n}$, are denoted by

$$\vec{v}^{\,n} = \begin{pmatrix} p^{n+\frac{1}{2}} \\ q^{n} \end{pmatrix} \qquad \tilde{e}^{\,n} = \vec{v}^{\,n} - \begin{pmatrix} p((n+\frac{1}{2})h) \\ q(nh) \end{pmatrix}$$

(d) For this method applied to the system of equations in **(a)** give a derivation of a relation between $||\tilde{e}^{\,n}||$ and $||\tilde{e}^{\,n-1}||$ that will lead to the establishment of a error bound for any $n \leq \mathrm{N}$. Indicate any constraints on the timestep must be satisfied in order for the relation to hold.

[6] (10 Pts.) Consider the convection diffusion equation

$$u_t + a\, u_x = b\, u_{xx}$$

with a and b positive constants and to be solved for $t > 0$, $0 \le x \le 1$, with periodic boundary conditions in x and smooth periodic initial data $u(x, 0) = f(x)$.

(a) For this problem construct a second order accurate, unconditionally stable and convergent finite difference scheme.

(b) How does your numerical solution behave as b goes down to 0?

Justify your answers.

[7] (10 Pts.) **(a)** Construct a convergent finite difference scheme to create approximate solutions to the system of equations

$$\begin{aligned} u_t &= u_x + v_x \\ v_t &= (1-a)v_x \end{aligned}$$

where the constant $a > 0$ and to be solved for $0 < x < 1$, with periodic boundary conditions, $u(0, t)$, $v(0, t)$ given and smooth.

(b) How does your numerical scheme behave as a goes down to 0?

Justify your answers

[8] (10 Pts.) Let Ω be an open, bounded and connected subset of R^2, with sufficiently smooth boundary. Consider the problem

$$-\frac{\partial}{\partial x}\Big((1 + 2x^2 + 3y^4)u_x\Big) - u_{yy} = f \text{ in } \Omega,$$

$$(1 + 2x^2 + 3y^4)u_x n_x + u_y n_y + \lambda u = g \text{ on } \Gamma = \partial\Omega,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\vec{n} = (n_x, n_y)$ is the outward unit normal to $\partial\Omega$, and $\lambda \ge 0$ is a constant.

(a) Give weak variational formulations of the problem by considering the cases $\lambda = 0$ and $\lambda > 0$. Show that each of these formulations have one and only one solution (under additional conditions on u, f or g, if necessary, that you will specify).

(b) In the case $\lambda > 0$, describe a FE approximation using P_1 elements, and a set of basis functions such that the corresponding linear system is sparse. In particular show that the corresponding finite dimensional problem has a unique solution.

(c) What would be a standard error estimate for (b) with P_1 elements function of the meshsize h ? (assuming convexity and sufficient regularity of Ω and of its boundary Γ).