Qualifying Exam, Fall 2023

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Consider $Ax = b$ with

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 7 & 2 \\ 1 & 2 & 4 \end{pmatrix},$$

and $b = (1, 9, -2)$.

(a) With $x_0 = (1, 1, 1)$, carry out one iteration of Gauss-Seidel method to find x_1.

(b) If we keep running the iterations, will the method converge? Why?

2. (10 points) Recall that the standard Conjugate Gradient algorithm can be described as

$$
\begin{aligned}
& r_0 = b - Ax_0, p_0 = r_0, \\
& \text{for } i = 0, 1, 2, \ldots \\
& \quad \alpha_i = (r_i^T r_i)/(p_i^T A p_i) \\
& \quad x_{i+1} = x_i + \alpha_i p_i \\
& \quad r_{i+1} = r_i - \alpha_i A p_i \\
& \quad \beta_i = (r_{i+1}^T r_{i+1})/(r_i^T r_i) \\
& \quad p_{i+1} = r_{i+1} + \beta_i p_i
\end{aligned}
$$

Show that CG for $Ax = b$ starting with x_0 is the same as applying the method to $Ay = r_0 = b - Ax_0$ starting with $y_0 = 0$, in the sense of producing the same iterates.

3. (10 points) Let $A \in \mathbb{R}^{n \times n}$ with entries $a_{i+1,i} = 1$ for $i = 1, \ldots, n-1$, $a_{1n} = 1$, and all other entries 0. Let b have entries $b_1 = 1$, $b_i = 0$ for $i = 2, \ldots, n$. Let x_0 be the zero vector. Prove that GMRES applies to $Ax = b$ with initial guess x_0

   (a) $\|b - Ax_k\| = 1$ for $1 \leq k \leq n-1$, and

   (b) takes n steps to find the true solution.

4. (10 points) Let A be Hermitian and tridiagonal and assume that the subdiagonal and superdiagonal entries of A are all nonzero.

   (a) Prove that all the eigenvalues of A must be distinct.

   (b) Prove that the matrix is diagonalizable.

5. (10 points) Assume A is such that $\|A\| = 1$. Recall there exist methods for numerically computing eigenvalues of A that compute exactly the eigenvalues of some perturbed matrix $A+\delta A$ with $\|\delta A\| = \mathcal{O}(\epsilon)$ (machine precision).

   (a) Prove that λ is an eigenvalue of $A + \delta A$ for some δA with $\|\delta A\|_2 \leq \varepsilon$, if and only if $\|(\lambda I - A)^{-1}\|_2 \geq 1/\varepsilon$.

   (b) Is it true that the eigenvalues numerically computed for A, that end up being the exact eigenvalues of some perturbed matrix $A + \delta A$ with $\|\delta A\| = \mathcal{O}(\epsilon)$, are close to the desired exact eigenvalues of A? Explain.

6. (10 points) Consider the singular value decomposition (SVD) of the matrix $A = U\Sigma V$, and consider the truncated SVD A_k obtained by extracting the upper left $k \times k$ submatrix of Σ (and appropriately resizing U and V). Prove that A_k is the best rank-k approximation of A in the Euclidean (spectral norm) sense, and that $\|A - A_k\| = \sigma_{k+1}$, where σ_{k+1} is the $(k+1)$th singular value of A.

7. (10 points) Consider the problem to find the extremizers of

$$x_1^2 + x_1 x_2 \quad \text{subject to} \quad x_1^2 \le x_2 \le 1.$$

Answer the following giving a complete reasoning for your answers:

(a) Write down the KKT conditions for this problem and find all points that satisfy them.

(b) Determine whether or not the points in part (a) satisfy the second order necessary conditions (SONC) for being local maximizers or minimizers.

(c) Determine whether or not the points that satisfy the SONC in part (b) satisfy the second order sufficient conditions (SOSC) for being local maximizers or minimizers.

8. (10 points) Recall that the *subdifferential* of a convex function f at x is defined as $\partial f(x) = \{g \in \mathbb{R}^n \colon f(y) \geq f(x) + \langle g, y - x \rangle \text{ for all } y \in \mathbb{R}^n\}$. Show the following:

   (a) If f is a convex, closed, proper function on $\mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $g(x) = f(Ax)$, then

   $$\partial g(x) \supseteq A^T \partial f(Ax) \quad \text{for all } x \in \mathbb{R}^n.$$

   (b) If f and g are convex, closed, proper functions on $\mathbb{R}^n$, then

   $$\partial (f + g)(x) \supseteq \partial f(x) + \partial g(x) \quad \text{for all } x \in \mathbb{R}^n.$$

   (c) When does equality hold in (a) and when does it hold in (b)?

9. (10 points) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex and differentiable function that satisfies $\|\nabla f(y) - \nabla f(x)\|_2 \le L\|y - x\|_2$ for any $x, y \in \mathbb{R}^n$, for some $L > 0$. Show that if we run gradient descent with fixed step size $\gamma \le 1/L$, then $O(1/\epsilon)$ iterations suffice to obtain an iterate $x^{(k)}$ with $f(x^{(k)}) - f(x^*) \le \epsilon$, where $f(x^*)$ is the optimum value.