Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1.

(1a) Let $(F_1; 0, 1, +_1, \times_1)$ and $(F_2; 0, 1, +_2, \times_2)$ be two fields sharing the zero and unit elements. Suppose that $(F_1 \cap F_2; 0, 1, +)$ is elementary in both $(F_1; 0, 1, +_1)$ and $(F_2; 0, 1, +_2)$. (In particular $+_1$ and $+_2$ agree on $F_1 \cap F_2$.) Prove that there is a structure $(H; 0, 1, +^*, \times_1^*, \times_2^*)$ so that $(F_1; 0, 1, +_1, \times_1) \preceq (H; 0, 1, +^*, \times_1^*)$ and $(F_2; 0, 1, +_2, \times_2) \preceq (H; 0, 1, +^*, \times_2^*)$.

(1b) Assuming there is $\times$ so that $(F_1 \cap F_2; 0, 1, +, \times)$ is elementary in both $(F_1; 0, 1, +_1, \times_1)$ and $(F_2; 0, 1, +_2, \times_2)$, prove there is a field $(H; 0^*, 1^*, +^*, \times^*)$ which elementarily extends both $(F_1; 0, 1, +_1, \times_1)$ and $(F_2; 0, 1, +_2, \times_2)$.

Problem 2. Let r_n, $n < \omega$ be a strictly increasing sequence of rational numbers, and consider the structure $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ where each r_i is named by a constant symbol.

Identify conditions on the sequence (r_n) which are equivalent to each of the following:

(2a) $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ is atomic.

(2b) $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ is saturated.

(2c) Every elementary submodel of $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ has a proper elementary submodel.

(2d) Every elementary submodel of $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ is isomorphic to $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$.

(2e) For every finite $a \subseteq \mathbb{Q}$, every elementary extension of $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ has a further elementary extension isomorphic to $(\mathbb{Q}; \leq, r_0, r_1, \ldots)$ by an isomorphism that fixes a.

Problem 3. Let T_0 and T_1 be computable consistent extensions of PA. (Note that $T_0 \cup T_1$ need not be consistent). Show that there is a sentence φ that is independent of T_0 and independent of T_1.

Problem 4. Let W_e be the eth computably enumerable subset of $\mathbb{N}$ according to a standard enumeration of c.e. sets. Let $\mathrm{FIN} := \{e \colon W_e \text{ is finite}\}$.

(4a) Show that FIN is Σ^0_2 complete.

(4b) Prove there is no partial computable function f from $\mathbb{N}$ to $\mathbb{N}$ so that if W_e is finite, then $f(e)$ halts and $|W_e| \leq f(e)$.

Problem 5. If $A \subseteq \mathbb{N}$, let A' be the Turing jump of A, and let $A^{(n)}$ be the nth iterate of the Turing jump relative to A.

(5a) Show that if $B \subseteq N$ and $\emptyset' \leq_m B$, then $\emptyset' \leq_1 B$.

(5b) Show that for all $n \geq 1$, if $B \subseteq N$ and $\emptyset^{(n)} \leq_m B$, then $\emptyset^{(n)} \leq_1 B$.

Problem 6. Assume the continuum hypothesis. Show that if T is a countable theory with infinite models, then T has a saturated model of size ω_1.

Problem 7. Prove that there is a collection S_ξ, $\xi < \omega_2$, of $\aleph_2$ many pairwise disjoint stationary subsets of ω_2.

Problem 8.

A *collapsing system* for θ is a collection of maps $\varphi_{\kappa,\alpha}$, for $\kappa < \theta$ a cardinal and for $\alpha < \kappa^+$, so that $\varphi_{\kappa,\alpha}$ is a surjection of κ onto α.

The system is *good* if for every $H \preceq V_\theta$ with transitive collapse map $\pi\colon H \to M$, if $\kappa \in H$ is a cardinal so that $\kappa \cap H$ is a cardinal τ, and if $\alpha \in [\kappa, \kappa^+) \cap H$, then $\varphi_{\tau,\pi(\alpha)} = \pi \circ (\varphi_{\kappa,\alpha} \restriction \tau)$.

Prove that if ZFC is consistent, the so is ZFC plus for every θ there is a good collapsing system.