Qualifying Exam, Fall 2020

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

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[1] (5 Pts.) Let $y > 0$ be a constant. Consider the iteration

$$p_{n+1} = \frac{1}{2}(p_n + \frac{y}{p_n}), \quad n \geq 0, \quad p_0 \text{ given.}$$

(a) Show that, if the sequence (p_n) converges to a limit p, then the sequence will approximate $p = \sqrt{y}$.

(b) Let $e_n = p_n - p = p_n - \sqrt{y}$ be the absolute error at step n. Express e_{n+1} function of e_n.

(c) Let $\hat{e}_n = \frac{e_n}{p} = \frac{p_n - p}{p}$ be the relative error. Using **(b)**, express $\hat{e}_{n+1}$ function of $\hat{e}_n$; then, assuming that $p_n > p = \sqrt{y}$, show convergence of the iteration.

[2] (5 Pts.) Suppose a value V is computed with a numerical procedure $\phi(h)$ and that $\lim_{h \to 0} \phi(h) = \mathrm{V}$. Assume there exists an asymptotic error expansion for $\phi(h)$ of the form

$$\mathrm{V} - \phi(h) \;=\; c_1\, h + c_2\, h^2 + c_3\, h^3 + \ldots$$

(a) How should the values $\phi(h)$ and $\phi(\frac{h}{3})$ be combined to yield an approximation to V that is $\mathrm{O}(h^2)$?

(b) How should the values $\phi(h)$, $\phi(\frac{h}{3})$ and $\phi(\frac{h}{2})$ be combined to yield an approximation to V that is $\mathrm{O}(h^3)$?

[3] (5 Pts.) Consider applying the fixed-point iteration to compute the solution of

$$\cos p = p.$$

Show that the iteration converges for any starting point p_0. You can use the following values: $\cos(-1) \approx 0.54030$ and $\sin(1) \approx 0.84147$.

[4] (5 Pts.) (a) Given $\{x_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ are the nodes and weights of an n-point Gauss quadrature rule for the integral approximation

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} f(x_i)\, w_i$$

derive the values of the nodes and weights $\{\tilde{x}_i\}_{i=1}^n$ and $\{\tilde{w}_i\}_{i=1}^n$ for the approximation

$$\int_{0}^{h} f(x)dx \approx \sum_{i=1}^{n} f(\tilde{x}_i)\, \tilde{w}_i \qquad \text{(I)}$$

(b) The n-point Gauss quadrature rule in **(b)** integrates polynomials up to degree $2n-1$ exactly. What is the order of accuracy with respect to h of the integral approximation (I)? Give a justification for your answer.

[5] (10 Pts.) (a) Give a derivation of the coefficients c_0, c_1 and c_2 so that the error in the approximation

$$y(t+h) \approx c_0\, y(t) \,+\, c_1\, y(t-h) \,+\, c_2\, y(t-2h)$$

has an order of accuracy with respect to h that is as high as possible.

(b) For the initial value problem

$$\frac{dy}{dt} = f(y), \quad y(t_0) = y_0.$$

consider the method

$$y^{k+1} = y^k + \frac{h}{2} f(y^k) + \frac{h}{2} f(c_0\, y^k \,+\, c_1\, y^{k-1} \,+\, c_2\, y^{k-2})$$

where the coefficients c_0, c_1 and c_2 are those derived in **(a)**. Derive an expression for the leading term of the local truncation error of this method.

(c) Consider the method applied to the model problem

$$\frac{dy}{dt} = \lambda\, y \quad y(0) = y_0.$$

where $\lambda \in$ C. Will the method in **(b)** converge to the solution? Justfiy your answer.

(d) Derive the conditions that determine the region of absolute stability for the method in **(b)**.

[6] (10 Pts.) Consider the system of equations

$$\begin{aligned} u_t + u_x + v_y &= 0 \\ v_t + v_x + u_y &= 0 \end{aligned}$$

to be solved for $t > 0$, $0 \leq x, y \leq 1$ with smooth initial data $u(x, y, 0)$ and $v(x, y, 0)$.

(a) What classes of constants $\mathrm{a, b, c, d, A, B, C, D}$ in the boundary conditions of the form

$$\begin{array}{rcrcll} \mathrm{a}\, u(0,y,t) & + & \mathrm{b}\, v(0,y,t) & = & 0 & \quad 0 \leq y \leq 1 \\ \mathrm{c}\, u(1,y,t) & + & \mathrm{d}\, v(1,y,t) & = & 0 & \quad 0 \leq y \leq 1 \\ & & & & & \\ \mathrm{A}\, u(x,0,t) & + & \mathrm{B}\, v(x,0,t) & = & 0 & \quad 0 \leq x \leq 1 \\ \mathrm{C}\, u(x,1,t) & + & \mathrm{D}\, v(x,1,t) & = & 0 & \quad 0 \leq x \leq 1 \end{array}$$

give a well posed problem?

(b) Devise a convergent finite difference scheme to create approximate solutions to this initial-boundary value problem when it is well-posed.

(c) Justify your answers.

[7](10 Pts.) Consider the scalar second order equation for $u(x, y, t)$,

$$u_t + uu_x + u^2 u_y = a\, u_{xx} \;+\; 2\, b\, u_{xy} \;+\; c\, u_{yy}$$

for a, b and c constants, to be solved for $t > 0$, $0 \leq x, y \leq 1$ with periodic boundary conditions at $x = 0$, $x = 1$, $y = 0$, $y = 1$ and smooth initial data

$$u(x, y, 0) = f(x, y).$$

(a) For what values of a, b and c is this problem well posed?

(b) Devise a convergent finite difference scheme to create approximate solutions to this problem when it is well-posed.

(c) Justify your answers.

[8] (10 Pts.) Consider the two dimensional problem,

$$-\triangle u = f \text{ in } \Omega,$$

$$\gamma u + \frac{\partial u}{\partial \vec{n}} = g \text{ on } \Gamma = \partial\Omega,$$

where Ω is an open, bounded, and connected subset of R^2 with a C^1 boundary $\Gamma = \partial\Omega$, and $\vec{n}$ the exterior unit normal to $\partial\Omega$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) State the necessary assumptions on the functions f and g, and on the constant γ, and prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution to the weak variational formulation.

(c) Setup a finite element approximation using P_1 elements, and a set of basis functions such that the associated linear system is sparse and of band structure. Show that the linear system has a unique solution, and give the rate of convergence for the approximation.