Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2000

MS: Do any 4 of the following 7 problems
Ph.D.: Do any 6 of the following 7 problems.

1. Consider the Dirichlet problem in a bounded domain $\mathcal{D} \subset R^N$ with smooth boundary S,

$$\Delta u + a(x)u = f(x), \qquad x \in \mathcal{D},$$
$$u_{|S} = 0, \qquad x \in S.$$

(a) Assuming that $|a(x)|$ is small enough, prove the uniqueness of the classical solution.

(b) Prove the existence of the solution in the Sobolev space $\dot{H}^1(\mathcal{D})$ assuming that $f \in L_2(\mathcal{D})$

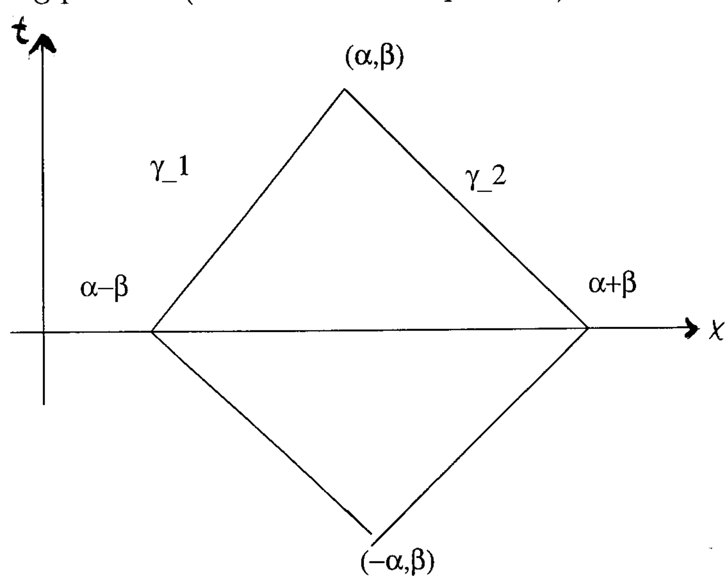
Note: Use Poincare inequality.

2. Consider the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + u^2(x,t) &= f(x,t), \quad x \in R^N, \ 0 < 0 < T, \\ u(x,0) &= 0. \end{aligned}$$

Prove the uniqueness of the classical bounded solution assuming that T is small enough.

3. Consider the following problem (so called Goursat problem):



find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x,t)u = 0$$

in the square $\mathcal{D}$, satisfying the boundary conditions

$$u_{|\gamma_1} = \varphi, \quad u_{|\gamma_2} = \psi,$$

where γ_1, γ_2 are two adjacent sides of $\mathcal{D}$. Here $a(x,t), \varphi$ and ψ are continuous functions. Prove the uniqueness of the solution of this Goursat problem.

4. Consider the following functional

$$F(v) = \int\int\int_{\mathcal{D}} \left[\sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k} \right)^2 + \alpha \left(\sum_{j=1}^{3} v_j^2(x) - 1 \right)^2 \right] dx,$$

where $x = (x_1, x_2, x_3) \in R^3$, $v(x) = (v_1(x), v_2(x), v_3(x))$, $\mathcal{D}$ is a bounded domain in R^3 with a smooth boundary S, and $\alpha > 0$ is a constant. Let $u(x) = (u_1(x), u_2(x), u_3(x))$ be the minimizer of $F(v)$ among all smooth functions satisfying the Dirichlet condition, $u_k(x) = \varphi_k(x)$, $k = 1, 2, 3$. Derive the system of differential equations that $u(x)$ satisfies.

5. Consider the eigenvalue problem on the interval $[0, 1]$,

$$-y''(t) + p(t)y(t) = \lambda y(t),$$
$$y(0) = y(1) = 0.$$

(a) Prove that all eigenvalues λ are simple.

(b) Prove that there is at most a finite number of negative eigenvalues.

6. Consider the initial boundary value problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) &= 0, \quad t > 0, x > 0, \\ u(x,0) &= 0, \quad x > 0 \\ u(0,t) &= g(t), \quad t > 0, \end{aligned}$$

where $g(t)$ is continuous function with a compact support, and a is constant. Find the explicit solution of this problem.

7. Consider the following system of ODEs

$$u_t = au - buv$$

$$v_t = -cv + duv$$

in which a, b, c, d are constants. For the phase plane region $R^{2+} = \{(u,v) : u > 0, v > 0\}$, do the following

a) Find all stationary points.

b) Analyze their type.

c) Draw a global picture of the solution set.

e) Show that R^{2+} is an invariant set for this flow.