Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Spring 2000

MS: Do any 4 of the following 8 problems

Ph.D.: Do any 6 of the following 8 problems.

1. Find the solution of $(\phi_x)^2+(\phi_y)^2 = 1$ in a neighborhood of the curve $y = x^2/2$ satisfying the conditions $\phi(x, x^2/2) = 0$ and $\phi_y(x, x^2/2) > 0$. Leave your answer in parametric form.

2. The equations of isotropic, linear elasticity for a homogeneous medium are

$$u_{tt} = (\lambda + \mu)\nabla(\nabla \cdot u) + \mu\Delta u,$$

where $u = (u_1, u_2, u_3)$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and λ and μ are positive constants. Use the *Ansatz*

$$u = e^{ik(\omega\cdot x-\alpha t)}(v_0(x,t) + v_1(x,t)/k + \cdots + v_N(x.t)/k^N),$$

where $|\omega| = 1$ and α and the v_j's to be determined, to construct asymptotic (as $k \to \infty$) solutions to the elastic wave equation travelling at the speeds $\sqrt{\lambda + 2\mu}$ and $\sqrt{\mu}$.

3. Consider the initial-boundary value problem for $u = u(x, y, t)$

$$u_t = \Delta u - u$$

for $(x, y) \in [0, 2\pi]^2$, with periodic boundary conditions and with

$$u(x, y, 0) = u_0(x, y)$$

in which u_0 is periodic. Find an asymptotic expansion for u for t large with terms tending to zero increasingly rapidly as $t \to \infty$.

4. a) Let (r, θ) be polar coordinates on the plane, i.e. $x_1 + ix_2 = r\exp(i\theta)$. Solve the boundary value problem $\Delta u = 0$ in $r < 1$, $\partial u/\partial r = f(\theta)$ on $r = 1$, beginning with the Fourier series for f (you may assume that f is continuously differentiable). Give your answer as a power series in $x_1 + ix_2$ plus a power series in $x_1 - ix_2$. There is a necessary

condition on f for this boundary value problem to be solvable that you will find in the course of doing this.

b) Sum the series in part a) to get a representation of u in the form

$$u(r,\theta) = \int_0^{2\pi} N(r, \theta - \theta')f(\theta')d\theta'.$$

5. Look for a traveling wave solution to the PDE

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

of the form $u(x,t) = v(x - ct)$. In particular, you should find an ODE for v. Under the assumption that v goes to a constant as $|x| \to \infty$, describe the form of the solution.

6. a) Consider the system of O.D.E.'s in R^{2n} given in vector notation by

$$\frac{dx}{dt} = f(|x|^2)p \text{ and } \frac{dp}{dt} = -f'(|x|^2)|p|^2 x,$$

where $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$, and f is a positive, smooth function on R. We use the notation $x \cdot p = x_1 p_1 + \cdots + x_n p_n$, $|x|^2 = x \cdot x$ and $|p|^2 = p \cdot p$. Show that $|x|$ is increasing with t when $p \cdot x > 0$ and decreasing with t when $p \cdot x < 0$, and that $H(x,p) = f(|x|^2)|p|^2$ is constant on solutions of the system.

(b) Suppose $f(s)/s$ has a critical value at $s = r^2$. Show that solutions with $x(0)$ on the sphere $|x| = r$ and $p(0)$ perpendicular to $x(0)$ must remain on the sphere $|x| = r$ for all t. [Compute $d(p \cdot x)/dt$ and use part a).]

7. Suppose that $u = u(x)$ for $x \in R^3$ is biharmonic; i.e. that $\Delta^2 u \equiv \Delta(\Delta u) = 0$. Show that

$$(4\pi r^2)^{-1} \int_{|x|=r} u(x)ds(x) = u(0) + (r^2/6)\Delta u(0)$$

through the following steps:

a) Show that for any smooth f,

$$(d/dr) \int_{|x| \le r} f(x)dx = \int_{|x|=r} f(x)ds(x)$$

b) Show that for any smooth f

$$(d/dr)(4\pi r^2)^{-1}\int_{|x|=r} f(x)ds(x) = (4\pi r^2)^{-1}\int_{|x|=r} n\cdot\nabla f(x,y)ds$$

in which n is the outward normal to the circle $|x| = r$.

c) Use step (b) to show that

$$(d/dr)(4\pi r^2)^{-1}\int_{|x|=r} f(x)ds(x) = (4\pi r^2)^{-1}\int_{|x|\le r} \Delta f(x)dx$$

d) Combine steps (a) and (c) to obtain the final result.

8. a) Show that for a smooth function f on the line, while $u(x,t) = f(t+|x|)/|x|$ may look like a solution of the wave equation $u_{tt} = \Delta u$ in three space dimensions, it actually is not. Do this by showing that for any smooth function $\phi(x,t)$ with compact support

$$\int_{R^3\times R} u(x,t)(\phi_{tt} - \Delta\phi)dxdt = 4\pi\int_R \phi(0,t)f(t)dt.$$

Note that, setting $r = |x|$, for any function function w which only depends on r one has $\Delta w = r^{-2}(r^2 w_r)_r = w_{rr} + \frac{2}{r}w_r$.

b) If $f(0) = f'(0) = 0$, what is the true solution to $u_{tt} = \Delta u$ with the initial conditions $u(x,0) = f(|x|)/|x|$ and $u_t(x,0) = f'(|x|)/|x|$?