Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2001

MS: Do any 4 of the following 7 problems
Ph.D.: Do any 6 of the following 7 problems.

1. Consider the initial value problem $u_t = a(u)$ with $u(0) = u_0$.
(i) Work out an example of the function $a(u)$ for which the solution u blows up in finite time.
(ii) Work out an example of the function $a(u)$ for which the solution u is not unique.
(ii) Describe conditions on the function $a(u)$ so that the solution u is unique and exists for all time. Justify your answer.

2. Consider the differential operator

$$L = (d/dx)^2 + 2(d/dx) + \alpha(x)u$$

in which α is a real-valued function. The domain is $x \in [0,1]$, with Neumann boundary conditions $du/dx(0) = du/dx(1) = 0$.
(i) Find a function $\phi = \phi(x)$ for which L is self-adjoint in the norm

$$||u||^2 = \int_0^1 u^2 \phi dx$$

.
(ii) Show that L must have a positive eigenvalue if α is not identically zeroand

$$\int_0^1 \alpha(x)dx \geq 0.$$

3. Let $u = u(x,t)$ solve the following PDE in three spatial dimensions

$$\Delta u = 0$$

for $R_1 < r < R(t)$, in which $r = |x|$ is the radial variable, with boundary conditions $u(r = R(t), t) = 0$ and $u(r = R_1, t) = 1$. In addition assume that $R(t)$ satisfies

$$dR/dt = -\partial u/\partial r(r = R)$$

with initial condition $R(0) = R_0$ in which $R_0 > R_1$.
(i) Find the solution $u(x,t)$.
(ii) Find an ODE for the outer radius $R(t)$.

4. For the ODE

$$\rho_{tt} = \rho(1-\rho)$$

do all of the following:
a) Analyze the type of all stationary points.
b) Find a conserved energy.
c) Draw a the phase plane diagram.

5. Consider the system

$$f_t + f_x = (h^2 - fg)$$

$$g_t - g_x = (h^2 - fg)$$

$$h_t = -(h^2 - fg)$$

a) Find two conserved quantities for this system.
b) Look for a traveling wave solution in which $(f, g, h) = (f(x-st), g(x-st), h(x-st))$, in which $|s| < 1$, and find a system of three ODEs for this special solution.
c) Reduce the system of ODEs for the traveling wave to a single ODE for h.
d) Show that the resulting ODE has solutions of the form

$$h = h_0 + h_1 tanh(\alpha x + x_0)$$

in which h_0, h_1, α and x_0 are constants.

6. Use the method of characteristics to solve the following partial differential equation in parametric form:

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 3u, \quad u(x,0) = u_0(x).$$

7. Consider the parabolic problem

$$u_t = u_{xx} + c(x)u$$

for $-\infty < x < \infty$, in which

$$c(x) = 0 \quad for \quad |x| > 1$$

$$c(x) = 1 \quad for \quad |x| < 1.$$

Find solutions of the form $u(x,t) = e^{\lambda t}v(x)$ in which $\int_{-\infty}^{\infty} |u|^2 dx < \infty$. (Hint: Look for v to have the form $a \exp -k|x|$ for $|x| > 1$ and $b \cos \ell x$ for $|x| < 1$ for some a, b, k, ℓ.)