Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Spring 2001
MS: Do any 4 of the following 7 problems
Ph.D.: Do any 6 of the following 7 problems.

1. For bodies (meaning bounded regions B in R^3) which are not perfectly conducting one considers the boundary value problem

$$0 = \nabla \cdot \gamma(x)\nabla u == \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\gamma(x)\frac{\partial u}{\partial x_j}\right)$$

with $u = f$ on ∂B, the boundary of B. The function $\gamma(x)$ is the "local conductivity" of B and u is the voltage. We define operator $\Lambda(f)$ mapping the boundary data f to the current density at the boundary by

$$\Lambda(f) = \gamma(x)\frac{\partial u}{\partial \nu},$$

and $\partial/\partial \nu$ is the inward normal derivative (this formula defines the current density).
a) Show that Λ is a symmetric operator, i.e. prove

$$\int_{\partial B} g\Lambda(f)dS = \int_{\partial B} f\Lambda(g)dS.$$

b) Use the positivity of $\gamma(x) > 0$ to show that Λ is negative a operator, i.e., prove

$$\int_{\partial B} f\Lambda(f)dS \leq 0.$$

2. a) Find the solution $u = (u_1(x,t), u_2(x,t))$, $(x,t) \in R \times R$, to the (strictly) hyperbolic equation

$$u_t - \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} u_x = 0,$$

satisfying $(u_1(x,0), u_2(x,0)) = (\exp(ixa), 0)$, $a \in R$.

b) Write the solution of initial value problem in part a) for general initial data

$$(u_1(x,0), u_2(x,0)) = (f(x), 0)$$

as an inverse Fourier transform. You may assume that f is smooth and rapidly decreasing as $|x| \to \infty$.

3. Solve the initial value problem

$$u_t = \frac{1}{2}((u_x)^2 + x^2), \quad u(x,0) = x.$$

You will find that the solution blows up in finite time. Explain this in terms of the characteristics for this equation.

4. The "Poincaré Inequality" states that for any bounded domain D in R^n there is a constant C such that

$$\int_D |u|^2 dx \leq C \int_D |\nabla u|^2 dx$$

for all smooth functions u which vanish on the boundary of D.

a) Find a formula for the "best" constant (that means the smallest one that works) for the domain D in terms of the eigenvalues of the Laplacian on D, and

b) give the best constant for the rectangular domain R^2

$$D = \{(x,y): \ 0 \leq x \leq a, \ 0 \leq y \leq b\}.$$

5. a) Show that the solution of the heat equation

$$u_t = u_{xx}, \qquad -\infty < x < \infty$$

with square-integrable initial data $u(x,0) = f(x)$, decays in time, and there is a constant α independent of f and t such that for all $t > 0$

$$\max_x |u_x(x,t)| \leq \alpha t^{-3/4} \left(\int_x |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

b) Consider the solution ρ of the transport equation $\rho_t + u\rho_x = 0$ with square-integrable initial data $\rho(x,0) = \rho_0(x)$ and the velocity u from part a). Show that $\rho(x,t)$ remains square- integrable for all finite time

$$\int_R |\rho(x,t)|^2 dx \leq e^{Ct^{1/4}} \int_R |\rho_0(x)|^2 dx,$$

where C does not depend on ρ_0.

6. a) Let B be a bounded region in R^3 with smooth boundary ∂B. The "conductor" potential for the body B is the solution of Laplace's equation outside B

$$\Delta V = 0 \text{ in } R^3 \backslash B$$

subject to the bounday conditions, $V = 1$ on ∂B and $V(x)$ tends to zero as $|x| \to \infty$. Assuming that the conductor potential exists, show that it is unique.

b) The "capacity" $C(B)$ of B is defined to be the limit of $|x|V(x)$ as $|x| \to \infty$. Show that

$$C(B) = \frac{-1}{4\pi} \int_{\partial B} \frac{\partial V}{\partial \nu} dS,$$

where ∂B is the boundary of B and ν is the outer unit normal to it (i.e. the normal pointing "toward infinity").

c) Suppose that $B' \subset B$. Show that $C(B') \leq C(B)$.

7. Consider the following system of PDEs:

$$f_t + f_x = g^2 - f^2$$

$$g_t - g_x = f^2 - g$$

a) Find a system of ODEs that describes traveling wave solutions of the PDE system; i.e. for solutions of the form $f(x, t) = f(x - st)$ and $g(x, t) = g(x - st)$.

b) Analyze the stationary points and draw the phase plane for this ODE system in the standing wave case $s = 0$.