Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2002

Each to the eight problems on this examination is worth 10 points, divided equally over sections of the problem except where indicated..

1. For the ODE

$$\begin{aligned} u_t &= u - v \\ v_t &= u(v^2 - 1) \end{aligned}$$

find and analyze the type of the stationary points and draw the phase plane diagram.

2. Consider the second order differential operator L defined by

$$Lu = -\frac{d^2u}{dx^2} + \epsilon x u$$

for $0 < x < \pi$ with boundary conditions

$$u(0) = u(\pi) = 0.$$

(a) (1 point) For $\epsilon = 0$ find the leading (i.e. smallest) eigenvalue λ_0 and the corresponding eigenfunction ϕ_0 for L.

(b) (9 points) For $\epsilon > 0$ look for the the eigenvalues and eigenfunctions to have an expansion of the form

$$\lambda = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2)$$
$$\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2)$$

Find formulas for λ_1 and ϕ_1 (your formulas will contain definite integrals which you do not need to evaluate).

3. Consider the first order system

$$\begin{aligned} u_t + u_x + v_x &= 0 \\ v_t + u_x - v_x &= 0 \end{aligned}$$

on the domain $0 < t < \infty$ and $0 < x < 1$. Which of the following sets of initial-boundary data are well posed for this system? Explain your answers.

(a) $u(x,0) = f(x)$, $v(x,0) = g(x)$

(b) $u(x,0) = f(x)$, $v(x,0) = g(x)$, $u(0,t) = h(x)$, $v(0,t) = k(x)$

(c) $u(x,0) = f(x)$, $v(x,0) = g(x)$, $u(0,t) = h(x)$, $v(1,t) = k(x)$

4. Consider the nonlinear hyperbolic equation $u_t + uu_x = 0$ for $-\infty < x < \infty$.

(a) Find a smooth solution to this equation for initial data of the form $u(x,0) = x$.

(b) Describe the breakdown of smoothness for the solution if $u(x,0) = -x$.

5. a) Suppose that u is a continuously differentiable function on $[0,1]$ with $u(0) = 0$. Starting with $u(x) = \int_0^x u'(t)dt$, prove the (sharp) estimate

$$\max_{[0,1]} |u(x)|^2 \leq \int_0^1 |u'(t)|^2 dt.$$

b) For any function p define $p_-(x) = -\min\{p(x), 0\}$. If p is continuous on $[0,2]$, show that all eigenvalues of

$$Lu = -u'' + pu \text{ on } [0,2] \qquad \int_0^2 2(t)\,dt \leq 1$$

with $u(0) = u(2) = 0$ are strictly postive if $\int_0^1 p_-(t)dt < 1$.

6. The temperature of a rod insulated at the ends with an exponentially decreasing heat source in it is a solution of the following boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-2t} g(x) \text{ for } (x,t) \in [0,1] \times \mathbf{R}_+,$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \text{ and } u(x,0) = f(x).$$

Find the solution to this problem by writing u as a cosine series,

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos n\pi x.$$

and determine $\lim_{t\to\infty} u(x,t)$.

7. For the right choice of the constant c, the function $F(x,y) = c(x+iy)^{-1}$ is a fundamental solution for the equation

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f \text{ in } \mathbf{R}^2$$

Find the right choice of c, and use your answer to compute the Fourier transform (in distribution sense) of $(x+iy)^{-1}$.

8. Let D be a bounded domain in $\mathbf{R}^3$ with smooth boundary ∂D. Show that a solution of the boundary value problem

$$\Delta^2 u = f \text{ in } D, \; u = \Delta u = 0 \text{ on } \partial D$$

must be unique.