Examination on Applied Differential Equations – Spring 2002

Do any four of the following problems for a Master's Pass. Do any six of the following problems for a Ph.D. pass.

Helpful Formula: If (r,θ) are polar coordinates on the plane, and $u(x(r,\theta),y(r,\theta)) = \tilde{u}(r,\theta)$, then

$$[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}](x(r,\theta),y(r,\theta)) = [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \tilde{u}}{\partial r})+\frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2}](r,\theta).$$

PROBLEMS

1.a) Find a radially symmetric solution, u, to the equation in $\mathbb{R}^2$,

$$\Delta u = \frac{1}{2\pi}\log|x|,$$

and show that u is a fundamental solution for Δ^2, i.e. show

$$\phi(0) = \int_{\mathbb{R}^2} u\Delta^2\phi dx$$

for any smooth ϕ which vanishes for $|x|$ large.

b) Explain how to construct the Green's function for the following boundary value in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D

$$w = 0 \text{ and } \frac{\partial w}{\partial n} = 0 \text{ on } \partial D,\ \ \Delta^2 w = f \text{ in } D,$$

where $\partial/\partial n$ denotes the normal derivative.

2 a) Given a continuous function f on $\mathbb{R}$ which vanishes for $|x| > R$, solve the initial value problem $u_{tt} - u_{xx} = f(x)\cos t,\ u(x,0) = u_t(x,0) = 0,\ -\infty < x < \infty,\ 0 \le t < \infty$ by first finding a particular solution by separation of variables and then adding the appropriate solution of the homogeneous PDE.

b) Since the particular solution is not unique, it will not be obvious that the solution to the initial value problem that you have found in part a) is unique. Prove that it is unique.

3. Steady viscous flow in a cylindrical pipe is described by the equation

$$(\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\Delta\vec{u} = 0$$

on the domain $-\infty < x_1 < \infty$, $x_2^2 + x_3^2 \le R^2$, where $\vec{u} = (u_1,u_2,u_3) = (U(x_2,x_3),0,0)$ is the velocity vector, $p(x_1,x_2,x_3)$ is the pressure, and η and ρ are constants.

a) Show that $\frac{\partial p}{\partial x_1}$ is a constant c, and that $\Delta U = c/\eta$.

b) Assuming further that U is radially symmetric and $U = 0$ on the surface of the pipe, determine the mass Q of fluid passing through a cross-section of pipe per unit time in terms of c, ρ, η and R. Note that

$$Q = \rho \int_{\{x_2^2 + x_3^2 \leq R^2\}} U dx_2 dx_3.$$

4. Use the Fourier transform on $L^2(\mathbb{R})$ to show that

$$\frac{du}{dx} + cu(x) + u(x-1) = f$$

has a unique solution $u \in L^2(\mathbb{R})$ for each $f \in L^2(\mathbb{R})$ when $|c| > 1$ – you may assume that c is a real number.

5. The following equation (called Fisher's equation) arises in the study of population genetics: $u_t = u(1-u) + u_{xx}$ on $-\infty < x < \infty$, $t > 0$. The solutions of physical interest satisfy $0 \leq u \leq 1$, and

$$\lim_{x \to -\infty} u(x,t) = 0, \quad \lim_{x \to \infty} u(x,t) = 1.$$

One class of solutions is the set of "wavefront" solutions. These have the form $u(x,t) = \phi(x+ct)$, $c \geq 0$.

Determine the ordinary differential equation and boundary conditions which ϕ must satisfy (to be of physical interest). Carry out a phase plane analysis of this equation, and show that physically interesting wavefront solutions are possible if $c \geq 2$, but not if $0 \leq c < 2$.

6. Consider the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 1$ in $\mathbb{R}^2$ with u prescribed on $y = 0$, i.e. $u(x,0) = f(x)$. Assuming that f is differentiable, what conditions on f insure that the problem is noncharacteristic? If f satisfies those conditions, show that the solution is

$$u(x,y) = f(r) - y + \frac{2y}{f'(r)}$$

where r must satisfy $y = (f'(r))^2(x-r)$. Finally, show that one can solve the equation for (x,y) in a sufficiently small neighborhood of $(x_0, 0)$ with $r(x_0, 0) = x_0$.

7. Consider the system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} \qquad (*)$$

Find an explicit solution for the following mixed problem for the system $(*)$:

$$(u(x,0), v(x,0)) = (f(x), 0) \text{ for } x > 0, \; u(t,0) = 0 \text{ for } t > 0.$$

You may assume that the function f is smooth and vanishes on a neighborhood of $x = 0$.

8. a) Assume that D is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D and outer unit normal ν. Find a variational formula for the lowest eigenvalue of $-\Delta u$ in D with the boundary condition $\frac{\partial u}{\partial \nu} + au = 0$ on ∂D, and show that the lowest eigenvalue will be positive or negative depending on the sign of a.

b) For the values of a which make the lowest eigenvalue positive, derive the following estimate for the solution u of the boundary value problem $-\Delta u + k^2 u = 0$ in $D, \frac{\partial u}{\partial \nu} + au = g$ on ∂D:

$$\max_D |u| \leq C_a \max_{\partial D} |g|,$$

where C_a does not depend on k. Use maximum principle arguments.