ADE W02

For a PhD pass do eight of the following problems.
For an MA pass do five of the following problems.

1. Consider the second order differential operator L defined by

$$Lu = \frac{d^2u}{dx^2} - u.$$

Find the the Green's function (= solution operator kernel) for the boundary value problem $Lu = f$ on $0 < x < 1$, $u(1) = u(0) = 0$.

2. a) Prove that

$$\int_0^\pi |u(x)|^2 dx \leq \int_0^\pi |\frac{du}{dx}|^2 dx$$

for all continuously differentiable functions u satisfying $u(0) = u(\pi) = 0$.

b) Consider the differential operator

$$Lu = -\frac{d^2u}{dx^2} + q(x)u, \; 0 < x < \pi$$

with the boundary conditions $u(0) = u(\pi) = 0$. Suppose q is continuous on $[0.\pi]$ and $q(x) > -1$ on $[0, \pi]$. Prove that all eigenvalues of L are positive.

3. For the differential equation $\rho_{tt} = \sin(\rho)$ do all of the following.

a) Find the stationary points and determine their type.

b) Find a conserved energy.

c) Draw the phase plane diagram.

4. a) Solve the initial value problem

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t)\frac{\partial u}{\partial x_k} + a_0(t)u = 0, \; x \in \mathbb{R}^n, \; u(0,x) = f(x)$$

where $a_k(t)$, $k = 1, \ldots, n$, and $a_0(t)$ are continuous functions, and f is a continuous function. You may assume f has compact support.

b) Solve the initial value problem

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(t) \frac{\partial u}{\partial x_k} + a_0(t)u = f(x,t), \ x \in \mathbb{R}^n, \ u(x,0) = 0,$$

where f is continuous in x and t.

5. Consider the boundary value problem

$$\Delta u + \sum_{k=1}^{n} \alpha_k \frac{\partial u}{\partial x_k} - u^3 = 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. If the α_k's are constants, and $u(x)$ has continuous derivatives up to second order, prove that u must vanish identically.

6. Solve the Cauchy problem

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0, \ t > 0, \ u(0,x) = 2 + x.$$

7. Consider the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) u = f \text{ in } \mathbb{R}^n, \tag{1}$$

where f is an integrable function (i.e. $f \in L^1(\mathbb{R}^n)$), satisfying $f(x) = 0$ for $|x| \geq R$. Solve (1) by Fourier transform, and prove the following results.

a) There is a solution of (1) belonging to $L^2(\mathbb{R}^n)$ if $n > 4$.

b) If $\int_{\mathbb{R}^n} f(x)dx = 0$, there is a solution of (1) belonging to $L^2(\mathbb{R}^n)$ if $n > 2$.

8. a) Find an explicit solution of the following Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t,x), \ u(0,x) = \frac{\partial u}{\partial x}(0,x) = 0.$$

b) Use part a) to prove the uniqueness of the solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(t,x)u = 0,\ u(0,x) = \frac{\partial u}{\partial x}(0,x) = 0.$$

Here $f(t,x)$ and $q(t,x)$ are continuous functions.

9. Let $\mathcal{D} = \{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$, and assume that f is continuous on $\mathcal{D}$ and vanishes for $|x| > R$.

a) Show that the boundary value problem

$$\Delta u = f \text{ in } \mathcal{D},\ u(x_1, 0) = \frac{\partial u}{\partial x_1}(0, x_2) = 0$$

can have only one bounded solution.

b) Find an explicit Green's function for this boundary value problem.