Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Winter 2003

Do all eight problems.

1. For the ODE

$$u_{tt} = u^3 - u \tag{1}$$

find and analyze the type of the stationary points and draw the phase plane diagram. Identify any connections between the stationary points, and any regions of periodic orbits.

2. Let L be the second order differential operator $L = \Delta - a(x)$ in which $x = (x_1, x_2, x_3)$ is in the three-dimensional cube $C = \{0 < x_i < 1,\ i = 1, 2, 3\}$. Suppose that $a > 0$ in C. Consider the eigenvalue problem

$$Lu = \lambda u \quad \text{for} \quad x \in C$$

$$u = 0 \quad \text{for} \quad x \in \partial C.$$

a) Show that all eigenvalues are negative.
b) If u and v are eigenfunctions for distinct eigenvalues λ and μ, show that u and v are orthogonal in the appropriate inner product.
c) If $a(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$ find an expression for the eigenvalues and eigenvectors of L in terms of the eigenvalues and eigenvectors of a set of one-dimensional problems.

3. Let Ω be a smooth domain in three dimensions and consider the initial-boundary value problem for the heat equation

$$u_t = \Delta u + f \quad \text{for} \quad x \in \Omega, \quad t > 0$$

$$\partial u / \partial n = 0 \quad \text{for} \quad x \in \partial\Omega, \quad t > 0$$

$$u = u_0 \quad \text{for} \quad x \in \Omega, \quad t = 0.$$

in which f and u_0 are known smooth functions with

$$\partial u_0 / \partial n = 0 \quad \text{for} \quad x \in \partial\Omega.$$

a) Find an approximate formula for u as $t \to \infty$.

b) If $u_0 \geq 0$ and $f > 0$, show that $u > 0$ for all $t > 0$.

4. 4. Consider the PDE

$$u_t = u_x + u^4 \quad \text{for} \quad t > 0$$

$$u = u_0 \quad \text{for} \quad t = 0$$

for $0 < x < 2\pi$. Define the set $A = \{u = u(x) : \hat{u}(k) = 0 \quad \text{if} \quad k < 0\}$, in which $\{\hat{u}(k,t)\}_{-\infty}^{\infty}$ is the Fourier series of u in x on $[0, 2\pi]$.
a) If $u_0 \in A$, show that $u(t) \in A$.
b) Find differential equations for $\hat{u}(0,t))$, $\hat{u}(1,t))$, and $\hat{u}(2,t))$.

5. Find a solution to $xu_x + (x+y)u_y = 1$ which satisfies $u(1,y) = y$ for $0 \leq y \leq 1$. Find the region in $\{x \geq 0,\ y \geq 0\}$ where u is uniquely determined by these conditions.
6. Assume that u is a harmonic function in the half ball $D = \{(x,y,z) : x^2 + y^2 + z^2 < 1,\ z \geq 0\}$ which is continuously differentiable, and satisfies $u(x,y,0) = 0$. Show that u can be extended to be a harmonic function in the whole ball. If you propose an explicit extension for u, be sure to explain why the extension is harmonic.

7. Under what conditions on g, continuous on $[0, L]$, is there a solution of

$$\frac{d^2u}{dx^2} = g,\ u(0) = u(L/3) = u(L) = 0?$$

8. a) Consider the damped wave equation for high-speed waves ($0 < \epsilon << 1$) in a bounded region D

$$\epsilon^2 u_{tt} + u_t = \Delta u$$

with the boundary condition $u(x,t) = 0$ on the boundary of D. Show that the energy functional

$$E(t) = \int_D \epsilon^2 u_t^2 + |\nabla u|^2 dx$$

is nonincreasing on solutions of the boundary value problem.
b) Consider the solution to the boundary value problem in part a) with initial data $u^\epsilon(x,0) = 0$, $u_t^\epsilon(x,0) = \epsilon^{-\alpha} f(x)$, where f does not depend on ϵ and $\alpha < 1$. Use part a) to show that

$$\int_D |\nabla u^\epsilon(x,t)|^2 dx \to 0$$

uniformly on $0 \leq t \leq T$ for any T as $\epsilon \to 0$.

c) Show that the result in part b) does not hold for $\alpha = 1$. To do this consider the case where f is an eigenfunction of the Laplacian, i.e. $\Delta f + \lambda f = 0$ in D and $f = 0$ on the boundary of D, and solve for u^ϵ explicitly.