Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Winter 2004

Solve any 7 of the following 9 problems.
Each problem has an equal value.

---

1) Consider the differential equation:

$$(1) \qquad \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + \lambda u(x,y) = 0$$

in the strip $\{(x,y),\ 0 < y < \pi,\ -\infty < x < +\infty\}$ with boundary conditions

$$(2) \qquad u(x,0) = 0, \quad u(x,\pi) = 0.$$

Find all bounded solution of the boundary value problem (1), (2) when

$$a)\ \ \lambda = 0, \qquad b)\ \ \lambda > 0, \qquad c)\ \ \lambda < 0$$

2) Let $C^2(\overline{\Omega})$ be the space of all twice continuously differentiable functions in the bounded smooth closed = domain $\overline{\Omega} \subset \mathbf{R}^2$. Let $u_0(x,y)$ be the function that minimizes the functional

$$D(u) = \int\int_\Omega \left[\left(\frac{\partial u(x,y)}{\partial = x} \right)^2 + \left(\frac{\partial u(x,y)}{\partial x} \right)^2 + f(x,y)u(x,y) \right] dxdy$$

$$+ \int_{\partial\Omega} a(s) u^2(x(s), y(s)) ds,$$

where $f(x,y)$ and $a(s)$ are given continuous functions and ds is the arclength element on $\partial\Omega$.

Find the differential equation and the boundary condition that u_0 satisfies.

3) Let $f(x_1, x_2)$ be a continuous function with compact support. Define

$$u(x_1, x_2) = \frac{1}{2\pi} \int\int_{\mathbf{R}^2} \frac{f(y_1, y_2) dy_1 dy_2}{z - w},$$

where $z = x_1 + ix_2,\ w = y_1 + iy_2$. Prove that

$$\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} = f(x_1, x_2) \quad \text{in} \quad \mathbf{R}^2.$$

4) Consider boundary value problem on $[0, \pi]$:

$$(1) \qquad -y''(x) + p(x)y(x) = f(x), \quad 0 < x < \pi,$$

$$(2) \qquad y(0) = 0, \quad y'(\pi) = 0.$$

Find the smallest λ_0 such that the boundary value problem (1), (2) has a unique solution whenever $p(x) > \lambda_0$ for all x. Justify your answer.

5) Consider the Laplace equation

$$(1) \qquad \frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0, \quad y > 0, \quad -\infty < x < +\infty$$

with the boundary condition

$$(2) \qquad \frac{\partial u(x,0)}{\partial y} - u(x,0) = f(x),$$

where $f(x) \in C_0^\infty(\mathbf{R}^1)$. Find a bounded solution $u(x,y)$ of (1), (2) and show that $u(x,y) \to 0$ when $|x| + y \to \infty$.

6) Consider the first order system $u_t - u_x = v_t + v_x = 0$ in the diamond shaped region $-1 < x + t < 1$, $-1 < x - t < 1$. For each of the following boundary value problems state whether this problem is well posed. If it is well-posed, find the solution.

(a) $u(x+t) = u_0(x+t)$ on $x - t = -1$, $v(x-t) = v_0(x-t)$ on $x + t = -1$

(a) $v(x+t) = v_0(x+t)$ on $x - t = -1$, $u(x-t) = u_0(x-t)$ on $x + t = -1$

7) For the two-point boundary value problem $Lf = f_{xx} - f$ on $-\infty < x < \infty$ with $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, the Green's function $G(x,x')$ solves $LG = \delta(x-x')$ in which L acts on the variable x.

(a) Show that $G(x,x') = G(x-x')$.

(b) For each x', show that

$$G(x,x') = \begin{cases} a_- e^{x} \text{ for } & x < x', \\ a_+ e^{-x} \text{ for } & x' < x, \end{cases}$$

in which $a_\pm$ are functions that depend only on x'.

(c) Using (a), find the x' dependence of $a_\pm$.

(d) Finish finding $G(x,x')$ by using the jump conditions to find the remaining unknowns in $a_\pm$.

8) For the ODE

$$\begin{aligned} (1) \qquad u_t &= u - v^2, \\ (2) \qquad v_t &= v - u^2 \end{aligned}$$

do all of the following:

a) Find all stationary points.

b) Analyze their type.

c) Show that $u = v$ is an invariant set for this ODE; i.e., if $u(0) = v(0)$, then $u(t) = v(t)$ for all t.

d) Draw the phase plane for this system.

9) Consider the initial value problem

$$u_{tt} = \Delta u$$

for $x \in R^d$ and $t > 0$, and with $u(x,0) = u_0(x), u_t(x,0) = u_1(x)$ in which $u_0(x) = u_1(x) = 0$ for $|x| < R_1$ and $|x| > R_2$. For $d = 2$ and $d = 3$, find the largest set $\Omega_0 \subset \{x \in R^d,\ t > 0\}$ on which $u = 0$ for any choice of u_0.