Ph. D. Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2005

(1) Consider the initial value problem

$$u_t = v, \quad v_t = |u|^\alpha,$$
$$u_{t=0} = u_0, \quad v_{t=0} = 0.$$

For what constant values of $u_0 \geq 0$ and $\alpha \geq 0$ is this problem well-posed, (a) only locally in time or (b) globally in time? Prove your answer.

(2) Consider the two point boundary value operator L defined for $u = u(x)$ by

$$Lu = u'' + u' - a(1 + x^2)u$$

defined on the interval $x \in [0, 1]$ with boundary conditions

$$u(0) = u(1) = 0$$

with $a > 0$. Let λ_{a0} be the eigenvalue of smallest absolute value for L and let u_{a0} be the corresponding eigenfunction. Do the following:

(a) Find an inner product in terms of which L is self-adjoint.
(b) Show that $\lambda_{a0} < 0$.
(c) Show that $|\lambda_{a0}|$ is an increasing function of a; i.e., if $0 < a_1 < a_2$, then $|\lambda_{a_1 0}| < |\lambda_{a_2 0}|$

(3) For the ODE $f'' - f(f^2 - 1)$ do the following:

(a) Find the stationary points and classify their type.
(b) Find all periodic orbits and all orbits that connect stationary points.
(c) Draw a picture of the phase plane.

(4) Consider the heat equation

$$u_t = u_{yy}$$

on the real line with initial data $u_0 = 1$, $y < 0$, $u_0 = 0$, $y > 0$. (a) Show that the solution $u(y, t)$ satisfies $\lim_{t\to\infty} u(y, t) = 1/2$. (b) Is the limit uniform in y? Prove your answer.

(5) The Cahn-Hilliard equation for phase separation of a binary alloy is

$$u_t + \Delta(\epsilon\Delta u - \frac{1}{\epsilon}W'(u)) = 0,$$

Where $W(u)$ is a smooth function of u. Show that

$$E(u) = \epsilon\frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{\epsilon}\int W(u)dx.$$

is a monotonically decreasing quantity for smooth solutions of the Cahn-Hilliard equation on the torus T^n.

(6) Let f be a smooth function defined on R^3 and suppose that $\Delta\Delta f = 0$ for $|x| \leq a$. Show that

$$(4\pi a^2)^{-1}\int_{|x|=a} f(x)ds = f(0) + \frac{a^2}{6}\Delta f(0).$$

Hint: Do this first for spherically symmetric f; i.e., for $f(x) = f(r = |x|)$, for which $\Delta = r^{-2}\partial_r(r^2\partial_r)$.

(7) Find the (entropy) solution for all time $t > 0$ of the inviscid Burgers equation $u_t + \frac{1}{2}(u^2)_x = 0$ with initial condition

$$u(x,0) = \begin{cases} 0, & x < -1 \\ x+1, & -1 < x < 0 \\ 1 - \frac{1}{2}x, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

(8) Consider the "eikonal" equation in R^2:

$$\phi_x^2 + \phi_y^2 = 1$$

in the domain $0 < x < 2\pi$ and $0 \leq y < \infty$, with periodic boundary conditions in x and boundary data

$$\phi(x,0) = \cos(x).$$

Find a solution in an implicit form.