Qualifying Examination on Applied Differential Equations

Wednesday, January 5 2005, 9.00 a.m.–1.00 p.m.

Solve all of the following 7 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1,\ t > 0, \tag{1}$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \quad t > 0,$$

and initial conditions

$$u(0,x) = e^{-x}\left(\pi\cos\pi x + \sin\pi x\right), \quad \frac{\partial u}{\partial t}(0,x) = 0, \quad 0 < x < 1.$$

- Show that a separation of variables in (1) leads to an eigenvalue problem in the variable x.
- Determine the eigenvalues and the eigenfunctions for the eigenvalue problem in question.
- Determine a solution to (1) which satisfies the boundary and the initial conditions.

Problem 2. Let $\varphi \in C^1(\mathbf{R}^2)$. Solve the following Cauchy problem in $\mathbf{R}^3$,

$$\begin{cases} x_1\partial_{x_1}u + 2x_2\partial_{x_2}u + \partial_{x_3}u = 3u, \\ u(x_1,x_2,0) = \varphi(x_1,x_2). \end{cases}$$

Problem 3. Let $u(x)$ be harmonic in the unit disc $|x| < 1$ in $\mathbf{R}^2$, and assume that $u \geq 0$. Prove the following *Harnack's inequality*:

$$\frac{1-|x|}{1+|x|}u(0) \leq u(x) \leq \frac{1+|x|}{1-|x|}u(0), \quad |x| < 1.$$

Problem 4. Let $u(x,t) \in C^\infty(\mathbf{R}^3 \times \mathbf{R})$ solve the Cauchy problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_x)\, u = 0, & x \in \mathbf{R}^3, \quad t > 0, \\ u|_{t=0} = \varphi(x), & \partial_t u|_{t=0} = \psi(x), \end{cases} \tag{2}$$

with $\varphi(x)$ and $\psi(x)$ being smooth compactly supported functions on $\mathbf{R}^3$. Use an explicit formula for the solution of (2) (the Kirchhoff's formula), to show that there exists a constant $C > 0$ such that we have, uniformly in $x \in \mathbf{R}^3$,

$$|u(x,t)| \le \frac{C}{t}, \quad t > 0.$$

Problem 5. Solve the inhomogeneous problem for the Laplace operator in the unit disc $\mathbf{D} = \{(x,y) \in \mathbf{R}^2; x^2 + y^2 < 1\}$,

$$\begin{cases} \Delta u = x^2 - y^2 \quad \text{in } \mathbf{D} \\ u = 0 \text{ along } \partial\mathbf{D}. \end{cases}$$

Problem 6. Find the Fourier transform of the integrable function $x \mapsto (\sin x)^2/x^2$.

Hint. Determine first the Fourier transform of $x \mapsto x^{-1}\sin x$.

Problem 7. Consider an autonomous system in $\mathbf{R}^n$, $x'(t) = f(x(t))$, where $f = (f_1, f_2, \dots, f_n)$ is a smooth vector field, such that

$$\sum_{k=1}^{n} x_k f_k(x) < 0 \quad \text{for } x \neq 0.$$

Show that $x(t) \to 0$ as $t \to \infty$, for each solution of the system, independently of the initial condition $x(0)$.