Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2006

Do all 8 problems. All problems are weighted equally.

1. Consider the second order ODE

$$x''(t) + x^3(t) - 4x(t) = 0. \tag{1}$$

   - Find the conserved quantity for (1)
   - Rewrite (1) as a first order system
   - Find and classify the equilibrium points
   - Sketch the phase portrait of the system

2. Consider the equation

$$u_{tt} = c^2 u_{xx} \tag{2}$$

   for $-at < x < at$ and $0 \le t$, in which a and c are positive constants. For which boundary conditions on $x = \pm at$ is there existence and uniqueness for this problem? Hint: The answer depends on a.

3. Consider the PDE

$$\begin{aligned} u_t &= \Delta u && (3) \\ u(x, y, t = 0) &= u_0(x, y) && (4) \end{aligned}$$

   in a half-plane $-\infty < x < \infty$ and $0 \le y < \infty$, with $u_0(x, y) \ge 0$. Compare the following two boundary conditions:

$$u(x, 0, t) = 0 \tag{5}$$

   and

$$u_y(x, 0, t) = 0. \tag{6}$$

   Denote the solution of (3), (4) and (5) as u^D and the solution of (3), (4) and (6) as u^N. Show that $u^D \le u^N$ for all x, y and $t > 0$.

4. Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0, \quad x \in \mathbf{R}, \tag{7}$$

together with the boundary condition

$$\frac{\partial u}{\partial y}(x,0) - u(x,0) = f(x), \tag{8}$$

where $f(x) \in C_0^\infty(\mathbf{R})$ (i.e., f is smooth with compact support). Find a representation for a bounded solution $u(x,y)$ of (7), (8), and show that $u(x,y) \to 0$ as $y \to \infty$, uniformly in $x \in \mathbf{R}$.

5. Let $a \in \mathbf{R}$ be a positive constant and $f(t)$ a non-negative continuous function. Assume that $y(t)$ is a continuous function such that

$$0 \le y(t) \le a + \int_0^t f(s) y^2(s)\, ds \quad \text{for } t \ge 0. \tag{9}$$

Show that

$$y(t) \le \frac{a}{1 - a\int_0^t f(s)\, ds}, \tag{10}$$

for all $t \ge 0$ for which the denominator in the right hand side of (10) is positive.

6. Let $\varphi \in C^1(\mathbf{C})$ be a function with compact support. When $\zeta \in \mathbf{C}$, let us write $\zeta = \xi + i\eta$, with ξ, $\eta \in \mathbf{R}$, and introduce the Cauchy-Riemann operator,

$$\frac{\partial}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right).$$

Let $z \in \mathbf{C}$. Show that

$$\varphi(z) = -\frac{1}{\pi}\iint \frac{\partial \varphi(\zeta)}{\partial \overline{\zeta}} (\zeta - z)^{-1}\, d\xi\, d\eta. \tag{11}$$

7. Let u solve the heat equation in a two-dimensional channel ; i.e.,

$$\begin{aligned} u_t &= \Delta u && (12)\\ u(x,y,t=0) &= u_0(x,y) && (13)\\ u_y(x,0,t=0) &= u_y(x,\pi,t=0) = 0 && (14) \end{aligned}$$

for $-\infty < x < \infty$ and $0 \leq y \leq \pi$. The initial data u_0 is assumed to be smooth and vanish for $|x|$ large.

(a) Show that $u(x, y, t)$ can be expanded in a cosine series in y; i.e.,

$$u(x, y, t) = \sum_{0 \leq k < \infty} \hat{u}(x, k, t) \cos(ky) \tag{15}$$

and find an equation for the k-th coefficient $\hat{u}(x, k, t)$.

(b) Find the limit of $t^{1/2} u(x, y, t)$ as $t \to \infty$.

8. Suppose that u is a smooth solution of the initial boundary value problem

$$u_t = u_{xx} + cu^2 \tag{16}$$

$$u(x, t = 0) = u_0(x) \tag{17}$$

$$u(0, t) = u(1, t) = 0 \tag{18}$$

for $0 < x < 1$, in which c is a positive constant.

(a) Show that

$$\frac{d}{dt} \int_0^1 |u(x,t)|^2 dx \leq -\left(\int_0^1 |u_x(x,t)|^2 dx \right) \left(1 - c \left(\int_0^1 |u(x,t)|^2 dx \right)^{1/2} \right).$$

Hint: First show that

$$\sup_x |u(x,t)|^2 \leq \int_0^1 |u_x(x,t)|^2 dx \tag{19}$$

(b) If the initial data u_0 satisfies $\int_0^1 |u_0(x)|^2 dx < 1/c^2$, show that u satisfies $\int_0^1 |u(x,t)|^2 dx < 1/c^2$ for all time. Hint: Show that $\frac{d}{dt} \int_0^1 |u(x,t)|^2 dx \leq 0$.

(c) If the boundary condition (18) is changed to $\partial_x u_0 = 0$ at $x = 0$ and $x = 1$, find a counterexample; i.e., find initial data u_0 for which the solution blows up in finite time.