Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2007

Please solve all 8 problems.

1. Let $\phi(x)$ be continuous and bounded in R^n. Assume that $\lim_{|x|\to\infty}\phi(x) = \phi_0$. Consider the Cauchy problem
$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) &= 0 \quad \text{for } 0 \le t,\, x \in R^n \\ u(x,0) &= \phi(x). \end{aligned}$$
Prove that $\lim_{t\to\infty} u(x,t) = \phi_0$.

2. Let $A_i(x)$, $i = 1, 2$, be smooth functions in a bounded domain $\Omega \subset R^n$ such that $A_1 = A_2$ on $\partial\Omega$. Assume that
$$\Delta A_1 + \sum_{j=1}^{n} \left(\frac{\partial A_1}{\partial x_j}\right)^2 = \Delta A_2 + \sum_{j=1}^{n} \left(\frac{\partial A_2}{\partial x_j}\right)^2$$
in Ω. Prove that $A_1(x) = A_2(x)$ in Ω.

3. Let S be a strip $\{0 < x_1 < a, -\infty < x_2 < \infty\}$. Let $u(x_1, x_2)$ be a smooth solution of $\Delta u + \lambda u = 0$ in S satisfying the boundary conditions $u(0, x_2) = 0$, $u(a, x_2) = 0$, $-\infty < x_2 < \infty\}$. Here λ is a real constant. Prove that if $\int_S |u(x_1, x_2)|^2 dx_1 dx_2 < \infty$, then $u(x_1, x_2) = 0$ in S.

4. Consider the initial boundary value problem:
$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\frac{\partial^2 u(x,t)}{\partial x \partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\frac{\partial u(x,t)}{\partial x} = 0 \qquad (1)$$
for $0 \le t < \infty$, $-\infty < x < \infty$ with
$$u(x,0) = f(x),\ \frac{\partial u(x,0)}{\partial t} = g(x) \qquad (2)$$
for $-\infty < x < \infty$, where $f(x), g(x)$ are smooth functions having compact supports and a is a smooth bounded function. Find an estimate for the solution of (1), (2) that will imply uniqueness.

5. Consider the initial value problem

$$\begin{aligned} du/dt &= cu^{1+\alpha} \\ u(0) &= u_0 \end{aligned}$$

in which $c > 0$ and $\alpha > 0$ are constants and $0 < u_0 < 1$.

(a) Find the solution of this *ODE*.

(b) Find the blowup time t_* at which $u \to \infty$.

(c) Find the value of α that minimizes t_* for fixed values of c and u_0.

6. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ in which $\mathbf{u} \in R^2$ and $\mathbf{x} \in R^2$. Solve the following problem by the method of characteristics

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{u} \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{x}. \end{aligned}$$

Note that the jth component of $\mathbf{u} \cdot \nabla \mathbf{u}$ is

$$(\mathbf{u} \cdot \nabla \mathbf{u})_j = \Sigma_{i=1}^2 u_i \partial_{x_i} u_j.$$

7. Let u and λ be the eigenfunction and eigenvalue of the two point boundary value problems on $0 \le x \le L$

$$\begin{aligned} u_{xx}(x) - a(x)u(x) &= -\lambda u(x) \\ u(0) = u(L) &= 0 \end{aligned} \qquad (3)$$

in which λ and L are constants. Assume that λ is the lowest eigenvalue for this problem

(a) Show that $a > 0$ implies $\lambda > 0$.

(b) Find an example showing that $a < 0$ does not imply $\lambda < 0$.

(c) Show that λ is a decreasing function of L.

8. For $i = 1, 2$ and $0 \le t \le T$, let $\Omega_i(t)$ be an open smooth bounded domain in R^2 for each t with $\Omega_1(0) = \Omega_2(0)$ and $\partial\Omega_1(t) \subset \Omega_2(t)$ for

$0 < t \leq T$ (i.e., $\Omega_1(t)$ is strictly contained in $\Omega_2(t)$ for $t = 0$). Let u_i for $i = 1, 2$ solve

$$\begin{aligned} \frac{\partial u_i}{\partial t} - \Delta u_i &= 0 \quad \text{for } x \in \Omega_i(t) \text{ and } 0 \leq t \leq T \\ u_i(x, 0) &= f(x) \quad \text{for } x \in \Omega_i(0) \\ u_i(x, t) &= 0 \quad \text{for } x \in \partial\Omega_i(t) \end{aligned}$$

in which the initial data f is independent of i with $f > 0$ in $\Omega_i(0)$.

(a) Show that $u_i > 0$ for $x \in \Omega_i(t)$ and $0 < t \leq T$

(b) Show that $u_1 < u_2$ for $x \in \Omega_1(t)$ and $0 < t \leq T$.