Applied Differential Equations, SPRING 2007

Each problem below is worth 10 points, so there are 80 points possible on this examination. Please start each problem on a new page.

1. Consider a minimizer u of the energy functional

$$E(u) = \frac{1}{2}\int (f-u)^2 dx + \frac{\lambda}{2}\int (\Delta u)^2 dx$$

where both u and f are periodic on the 2-torus. The above energy represents a least squares fit to the data f while having a relatively small size for the 'bending' energy of u, represented by the L^2 norm of the Laplacian.

(a) Show that the Euler-Lagrange equation for u is $-(f-u)+\lambda\Delta^2 u = 0$.

(b) Compute a solution of this problem in terms of a Fourier series expansion.

(c) Discuss how the high frequency modes depend on the value of λ which imparts some smoothing to u.

2. Find all solutions to the boundary value problem $\Delta u = x$ in $x^2+y^2<1$, $\partial u/\partial r = y$ on $x^2+y^2=1$. Polar coordinates are useful here. In polar coordinates

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

3. Consider the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y \qquad (*)$$

in the domain $y>0$, $-\infty<x<\infty$, $-\infty<t<\infty$. Find

i) all boundary conditions of the form $a(x)u(x,0)+b(x)v(x,0)=0$ such that the energy

$$E(t) = \int_{\{y>0\}} u^2(x,y,t)+v^2(x,y,t)\,dxdy$$

remains constant for solutions of $(*)$, and

ii) find all boundary conditions of that form for which $E(t)$ does not increase as $t\to\infty$.

4. Suppose that $\Delta u = 0$ in a bounded domain D and that $u\in C^3(\overline{D})$. Show that $|\nabla u|^2$ takes its maximum value in $\overline{D}$ on the boundary of D. [Consider $\Delta(|\nabla u|^2)$.]

5. Consider the equation

$$u_t + (u^2)_x = au^2,$$

with $a>0$ and with initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } |x|>1 \\ 1+x & \text{if } -1<x<0 \\ 1-x & \text{if } 0<x<1 \end{cases}$$

(a) Solve this problem by the method of characteristics to get functions $w(y,t)$ and $x(y,t)$ such that the solution $u(x,t)$ must satisfy $u(x(y,t),t) = w(y,t)$. To really find $u(x,t)$ you would have to solve $x = x(y,t)$ for $y(x,t)$, but do not attempt to do that.

(b) The functions $w(y,t)$ and $x(y,t)$ will not exist for all $t \geq 0$ and $y \in \mathbb{R}$. Find t^*, the largest number such that $w(y,t)$ is finite for $0 \leq t < t^*$ for all $y \in \mathbb{R}$.

(c) Will it be possible to solve $x = x(y,t)$ for $y(x,t)$ for all t in the interval $[0, t^*)$? Explain your answer.

6. Consider the fourth order ODE

$$-Cu' + (u^3 - u^2)' = -u''''. \tag{**}$$

(a) We are looking for solutions to (**) which tend to limits u_l as $x \to -\infty$ and u_r as $x \to +\infty$ with $u_l \neq u_r$. Assuming that such a solution exists, find the value of C.

(b) For solutions of the form described above, integrate the equation and write it as a third order equation. Determine the constant of integration in terms of u_l and u_r.

(c) Write the solution of (b) as a first order system of three equations and identify all equilibria.

(d) Determine the dimensions of the stable and unstable manifolds at the equilibria, i.e. find the dimensions of the sets of solutions near each equilibrium which converge to the equilibrium as $x \to \infty$ and $x \to -\infty$ respectively.

7. a) Suppose that $a(\alpha)$ is a smooth function (continuous derivatives of all orders) which vanishes for $|\alpha| > R$. If the derivative of $\phi(\alpha)$ does not vanish for $|\alpha| \leq R$, show that

$$F(k) = \int_{\mathbb{R}} e^{ik\phi(\alpha)} a(\alpha) d\alpha$$

satisfies $|F(k)| \leq C_N k^{-N}$ for all N for some sequence of constants C_N.

b) Consider the solution to $\Delta u + k^2 u = 0$ given by

$$u(x,y,k) = \int e^{ik(x \sin\alpha - y\cos\alpha - \alpha)} a(\alpha) d\alpha,$$

where $a(\alpha)$ is as in part a). Show that $|u(x,y,k)| \leq C_N k^{-N}$ for all N on $x^2+y^2 < 1$.

c) Suppose that $a(\alpha) = 0$ for $|\alpha| > \pi$. Show that

$$u(1,0,k) = \frac{a(0)}{k^{1/3}} \int_{\mathbb{R}} e^{-i\eta^3/6} d\eta + O(k^{-2/3})$$

as $k \to \infty$.

8. The porous media equation in $\mathbb{R}^n$ is

$$u_t = \Delta u^m, \quad m > 1.$$

Consider a similarity solution of the form $t^{-\alpha}U(x/t^{\beta})$ where U is nonnegative.

(a) Compute the values of α and β depending on the dimension of space (hint: the PDE conserves $\int u(x,t)dx$).

(b) Show that $U(\eta)$ satisfies an elliptic PDE of the form

$$C_1 U + C_2 \eta \cdot \nabla U + \Delta(U^m) = 0.$$

Compute C_1 and C_2 in terms of α and β.

(c) Find a family of radially symmetric solutions of the PDE in (b). Use the fact that for radially symmetric $f(r)$, $\nabla f = f_r \hat{r}$ and $\Delta f = f_{rr} + \frac{n-1}{r} f_r$, where $\hat{r}$ is the unit vector pointing outward from the origin, and n is dimension of space.

(d) Find the special solution with unit mass, $\int u(x,t)dx = 1$.