Applied Differential Equations – Fall 2008

Each problem is worth 10 points.

1. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. Let f be a continuous function in Ω and g be a continuous function on Γ.

(a) Consider the functional

$$J[u] = \int_\Omega (|Du|^2 + fu)dx + \int_\Gamma gu^2 d\sigma$$

applied to smooth functions u defined in $\Omega \cup \Gamma$. Determine the differential equation and boundary condition satisfied by a function which minimizes J.

(b) State and prove conditions for uniqueness of solutions to the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u_\nu + gu = 1 \text{ in } \Gamma,$$

where ν denotes the normal vector at each point on Γ, outward with respect to Ω.

2. Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function with $g(0) = 0$. Derive an integral formula for the solution of the problem

$$u_t - u_{xx} = 0 \text{ in } \mathbb{R}^+ \times (0,\infty),\ u = 0 \text{ on } \mathbb{R}^+ \times \{t = 0\},\ \text{and } u = g \text{ on } \{x = 0\} \times [0,\infty)$$

in terms of g. Consider the function $v(x,t) = u(x,t) - g(t)$ extended to $\mathbb{R} \times \mathbb{R}^+$ by $v(x,t) = -v(-x,t)$.

3. Consider the initial value problem for Burger's equation

$$u_t + u_x u = 0 \text{ in } \mathbb{R} \times (0,\infty),\ u = g \text{ on } \mathbb{R} \times \{t = 0\}.$$

Find the entropy solution of this problem with the initial data

$$g(x) = \begin{cases} 0 & \text{if } x > 1, \\ 1 - x & \text{if } 0 < x < 1, \\ 1 & \text{if } x < 0, \end{cases}$$

Also find the maximal time interval $[0, t^*)$ on which the solution is continuous.

4. A traveling wave solution of speed c to $u_t = u_{xx} + 1 - u^2$ is a solution of the form $u(x,t) = f(x - ct)$. Using phase plane analysis, explain how this equation has a unique traveling wave solution of speed c with $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = -1$ as long as $c > 0$. A rigorous argument is not asked for here. Then prove that the function $f(x)$ will not be monotonic decreasing when $c < 2\sqrt{2}$.

5. Suppose that $f(x)$ is a continuous function such that $f(x) \equiv 0$ when $|x| > R$. Show that

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$

is a 'weak solution' to $\Delta u = f$ in the sense that

$$\int u \Delta \phi dx = \int f \phi dx$$

for all $\phi \in C^2(\mathbb{R}^3)$ satisfying $\phi(x) \equiv 0$ for $|x| > R + 1$.

6. Consider the first order system of equations

$$u_t + \sum_{j=1}^{n} A_j u_{x_j} = 0, \tag{1}$$

where $u(x,t) = (u_1(x,t), .., u_m(x,t))$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, and the A_j's are symmetric $m \times m$ matrices with constant real entries. Use an energy argument to show that the domain of dependence of (x_0, t_0), $t_0 > 0$, for a solution of the system (1) is contained in the cone

$$\{|x - x_0| \leq \Lambda(t_0 - t)\}$$

where $\Lambda = \max_{\{|\xi|=1, 1 \leq l \leq m\}} |\lambda_l(\xi)|$, and $\lambda_l(\xi)$, $l = 1, .., m$, are the eigenvalues of the matrix $A(\xi) = \sum_{j=1}^{n} \xi_j A_j$.

7. Suppose u is a smooth solution of the following problem

$$u_{xxt} + u_{xx} - u^3 = 0 \text{ in } [0,1] \times (0,\infty), \; u(0,t) = u(1,t) = 0 \text{ for } (0,\infty)$$

with initial data $u(x,0) = x(x-1)$. Derive a differential inequality for $w(t) := \int_0^1 (u_x)^2(x,t)dx$, and show that $u(x,t)$ uniformly tends to zero as $t \to \infty$.

8. Suppose that $q(x)$ is a real-valued continuous function such that $\int_0^1 q(x)dx = 0$, but $q(x)$ is not identically zero. Show that $Lu = -u'' + q(x)u$ with the boundary conditions $u'(0) = u'(1) = 0$ must have a strictly negative eigenvalue by showing that $\int_0^1 uLudx$ can be negative.