ADE Qual Spring 2008

1. Consider the eigenvalue problem

$$y'' + \lambda y = 0, \quad x \in (0, \ell)$$
$$y'(\ell) + y(\ell) = 0$$
$$y(0) = 0$$

(a) Show that if f and g satisfy the boundary conditions, that $(f'(x)g(x) - g'(x)f(x))|_{x=0}^{x=\ell}=0$.

(b) Use this property of the boundary conditions to prove that all eigenfunctions u_1 and u_2 are orthogonal in the L^2 sense

$$\int_0^\ell u_1 u_2 dx = 0.$$

(c) Find an equation satisfied by the eigenvalues, and find the corresponding eigenfunctions. (Note: You may not be able to find an explicit formula for these eigenvalues.) Show graphically that there are an infinite number of positive eigenvalues $\{\lambda_n\}$ with $\lambda_n \to +\infty$.

2. Use the method of characteristics to solve the Eikonal equation $(u_x)^2 + (u_y)^2 = 1$ with initial values $u|_\Gamma = 1$ on the unit circle $\Gamma = \{(x, y)|x^2 + y^2 = 1\}$.

3. Use energy methods to prove uniqueness of solutions to the following IVP, assuming that the initial data has compact support

$$\mathbf{u_t} + \Sigma_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{A_i u_{x_i}} = \mathbf{0}.$$

Here, $\mathbf{u} : \Re^{\mathbf{n}} \times [\mathbf{0}, \infty) \to \Re^{\mathbf{m}}$, $\mathbf{A_i} \in \Re^{\mathbf{m\times m}}$ and $\mathbf{A_i}$ are constant and symmetric.

4. Solve the following IVP

$$u_{tt} + u_{xt} - 20u_{xx} = 0$$
$$u(x, 0) = \phi(x)$$
$$u_t(x, 0) = \psi(x).$$

5. Let $K_a(x-y)$ and $K_b(x-y)$ be the kernels of the operators $(\Delta - aI)^{-1}$ and $(\Delta - bI)^{-1}$ on $L^2(R^n)$, where $0 < a < b$. Show that $(\Delta - aI)(\Delta - bI)$ has a fundamental solution of the form $c_1 K_a + c_2 K_b$. Compute the constants c_1 and c_2 in terms of a and b.

6. Consider the differential equation

$$u_t = -\epsilon \Delta u + \Delta^3 u$$

on the interval $[0, 2\pi]$ with periodic boundary conditions. Find the largest value of ϵ_0 so that the solution of the PDE always stays bounded as $t \to \infty$, if $\epsilon < \epsilon_0$. Justify your answer.

7. Consider the spatially dependent logistic equation

$$u_t = \Delta u + \beta u(1-u)$$

on the N-torus T^N.

(a) Consider a smooth solution with positive bounded initial data. Prove the solution stays positive on any time interval of existence.

(b) Derive an a priori upper bound for the solution from (a) depending on the maximum of the initial data $M = max_x u(\cdot, 0)$. Hint: consider separately the case $M > 1$ and $0 < M < 1$.

8. Consider the phase plane problem

$$\begin{aligned} \dot{x} &= -aH_y(x,y) - bH_x(x,y), \\ \dot{y} &= aH_x(x,y) - bH_y(x,y), \end{aligned}$$

where H is a smooth function in the plane with a local minimum at the origin.

(a) for $a = 0$ and $b > 0$ prove that $(0,0)$ is a stable equilibrium of the dynamical system.

(b) for $a > 0$ and $b = 0$ prove that the function H is conserved along any forward or backward time trajectory.

(c) Consider the case $b > 0$ and assume that H has a local minimum with the Hessian matrix H positive definite at the origin. Prove that there exists a neighborhood of $(0,0)$ such that all forward time trajectories converge towards the origin. Recall that the Hessian matrix is

$$\begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix}.$$