Qualifying Exam on Applied Differential Equations

Tuesday, September 15, 2009, 2:00 p.m.–6:00 p.m.

Solve the following 8 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Let $u(x)$ be harmonic in the open ball $\{x \in \mathbf{R}^n; |x| < R\}$. Assume that $u(x) \geq 0$. Show that the following *Harnack inequality* holds,

$$\frac{R^2 - |x|^2}{(R + |x|)^n} u(0) \leq R^{2-n} u(x) \leq \frac{R^2 - |x|^2}{(R - |x|)^n} u(0), \quad |x| < R.$$

Problem 2. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $V \in C(\overline{\Omega})$. Show that for $\varepsilon > 0$ small enough, the Dirichlet problem

$$(-\Delta + \varepsilon V)\, u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{along} \quad \partial\Omega$$

has a unique solution in the space $H_0^1(\Omega)$, for each $f \in L^2(\Omega)$.

Problem 3. Consider the equation (sometimes called the Beltrami equation)

$$\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} - \mu\left(\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2}\right) = \frac{\partial g}{\partial x_1} + i\frac{\partial g}{\partial x_2}.$$

For each smooth function g of compact support this equation is supposed to determine a smooth square-integrable function u. Show that it does that provided that the complex number μ satisfies $|\mu| \neq 1$, and one has the estimate

$$||\, u \,||_{L^2(\mathbf{R}^2)} \leq \frac{1}{||\mu| - 1|} ||\, g \,||_{L^2(\mathbf{R}^2)}.$$

Problem 4. Let $Lu = -u_{xx} + V(x)u$, where $V(x)$ is real-valued, and $Au = 4u_{xxx} - 3((Vu)_x + Vu_x)$. A page of exciting computations shows that the commutator $LA - AL$ is given by

$$(LA - AL)u = (6VV_x - V_{xxx})u.$$

Do not do that computation during this examination. Instead suppose that V depends on the parameter t as well as x, and is a solution of the evolution equation $V_t = 6VV_x - V_{xxx}$ (the Korteweg-De Vries equation). Suppose that $u(x,t)$ satisfies

$$L(t)u = -u_{xx} + V(t)u = \lambda(t)u \text{ and } \int_{\mathbf{R}} u^2(x,t)dx \equiv 1,$$

i.e. $u(x,t)$ is a normalized eigenfunction for the operator $L(t)$. Show that $\lambda(t)$ must be independent of t.

Problem 5. Solve the Hamilton-Jacobi equation,

$$\begin{cases} u_t + \frac{1}{2}(u_x)^2 - x = 0, \\ u(x,0) = \alpha x, \quad \alpha \in \mathbf{R}. \end{cases}$$

The solution is linear in x, but we want you to use the method of characteristics to solve this problem. The linearity in x is a check on your answer.

Problem 6. Let $x(t)$ be a nonnegative differentiable function such that

$$x'(t) \geq \frac{1}{1 + tx(t)} + t - 1, \quad t \geq 0.$$

Show that $x(t) \geq 1 - \exp(-t^2/2)$ for $t \geq 0$.

Hint. Derive a differential equation for the function $t \mapsto 1 - \exp(-t^2/2)$.

Problem 7. Let $u(x,t)$ solve the wave equation

$$\begin{cases} (\partial_t^2 - \Delta) u(x,t) = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \\ u(x,0) = \varphi \in C_0^\infty(\mathbf{R}^n), \quad \partial_t u(x,0) = 0. \end{cases}$$

Show that the function

$$\widetilde{u}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} u(x,s)\, ds, \quad t > 0$$

satisfies the initial value problem for the heat equation,

$$(\partial_t - \Delta)\, \widetilde{u}(x,t) = 0, \quad t > 0, \quad \widetilde{u}(x,0) = \varphi(x).$$

This is sometimes called the *transmutation formula*.

Problem 8. Consider a linear damped wave equation with a constant damping factor $a \in (0,1)$,

$$\begin{cases} (\partial_t^2 - \Delta_x + a\partial_t)\, u(x,t) = 0, \\ u(x,0) = 0, \quad \partial_t u(x,0) = f(x). \end{cases}$$

Here $t \geq 0$ and $x \in \mathbf{T}^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$. Assume also that $f \in C^\infty(\mathbf{T}^2)$.

- Find an explicit formula for the solution of this problem $u(x,t)$ in terms of Fourier series.

- Show that the energy of the solution,

$$E(t) = \frac{1}{2} \int_{\mathbf{T}^2} \left(|\nabla_x u|^2 + |\partial_t u|^2 \right) dx, \quad t \geq 0$$

decreases at an exponential rate as $t \to \infty$. What is the rate of the exponential decay?