Qualifying Exam on Applied Differential Equations

Wednesday, March 25, 2009, 2:00 p.m.–6:00 p.m.

Solve any 7 of the following 9 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Let $h(t)$ and $a(t)$ be continuous and bounded functions on $[0, \infty)$, with $a(t) \geq 0$. Let $x(t)$ be a continuous function such that

$$x(t) \leq h(t) \int_0^t a(s)x(s)\, ds + \frac{1}{1+t^2}, \quad t \geq 0.$$

Assume that

$$\int_0^\infty |h(t)|\, dt < \infty.$$

Show that $x(t)$ is bounded above on $[0, \infty)$.

Problem 2. Let $p(x) \in C^1([0,1])$ and $q \in C([0,1])$ be real-valued with $p(x) > 0$. Show that the eigenvalue problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda u, \quad u(0) = u(1) = 0$$

has the following properties:

- All eigenvalues are simple.
- There are at most finitely many negative eigenvalues.

Problem 3. Let $\Omega \subset \mathbf{R}^n$ be an open set and let $u \in C^\infty(\Omega)$ be harmonic in Ω, so that

$$\Delta u = 0.$$

Show that there exists a constant $C = C(n)$ depending on the dimension n only such that

$$|\nabla u(x)| \leq \frac{C}{d(x)} \sup_\Omega |u(x)|, \quad x \in \Omega. \tag{1}$$

Here

$$d(x) = \inf_{y \in \partial\Omega} |x - y|$$

is the Euclidean distance from x to the boundary of Ω. Generalize (1) to obtain similar bounds on the higher order derivatives of u.

Hint. Use the Poisson formula for the function u in a ball.

Problem 4. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $V \in C(\overline{\Omega})$ satisfy $V(x) \geq 0$. Show that for each $f \in L^2(\Omega)$, the Dirichlet problem

$$(-\Delta + V)\, u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{along} \quad \partial\Omega$$

has a unique solution in the space $H_0^1(\Omega)$.

Problem 5. Consider the complementary error function

$$F(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.$$

Show that we have, as $x \to \infty$,

$$F(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right).$$

Show also that this estimate for large x can be refined to a complete asymptotic expansion,

$$F(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{a_k}{x^{2k}},$$

for some coefficients a_k. (You do not have to determine the a_k's).

Problem 6. Consider an initial value problem for the focusing cubic non-linear Schrödinger equation,

$$iu_t = -\frac{1}{2} u_{xx} - |u|^2 u, \quad u(x,0) = \varphi(x). \tag{2}$$

Show that the following are conserved quantities for (2) (you may assume that the function $u(x,t)$ vanishes as $|x| \to \infty$, together with all of its derivatives).

- Mass

$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx$$

- Energy

$$\int_{-\infty}^{\infty} \left(\frac{1}{2} |\partial_x u(x,t)|^2 - \frac{1}{2} |u(x,t)|^4\right) dx.$$

Hint. The function $u(x,t)$ in (2) is complex-valued. In the computations, use that $|u|^2 = u\overline{u}$.

Problem 7. Solve the following PDE,

$$\begin{cases} u_t + u_x^2 = 0, \\ u(x,0) = -x^2. \end{cases}$$

Find the time T for which $|u| \to \infty$ as $t \to T$.

Problem 8. Consider the hyperbolic equation

$$u_{tt} + 3u_{xt} + 2u_{xx} = 0 \tag{3}$$

in the quarter-plane $Q = \{x > 0, t > 0\}$. Assign boundary conditions along $t = 0$ and $x = 0$ such that the boundary value problem for (3) in Q will have a unique solution.

Problem 9. Consider the boundary value problem in a smooth bounded domain D in $\mathbf{R}^n$,

$$\Delta u = 0 \quad \text{in } D, \quad \frac{\partial u}{\partial n} + a(x)u = f \quad \text{on the boundary of } D.$$

Here n is outer unit normal to ∂D.

- Find a functional whose Euler-Lagrange equation leads to the boundary value problem above.
- Assume that $a(x) > 0$. Prove that this boundary value problem has a unique smooth solution.