Applied Differential Equations – Fall 2010

1. Determine the constants A such that the differential equation

$$\frac{d^2u}{dx^2} + u = A + x$$

has a solution satisfying $u(0) = u(\pi) = 0$.

2. (a) Solve

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

with the initial data $(u(x,0), v(x,0)) = (f(x), g(x))$

(b) Find all boundary conditions of the form $au(0,t) + bv(0,t) = 0$ which make the initial value problem in part (a) well-posed in $x \geq 0,\ t \geq 0$.

3. Consider the competition with limited resources model

$$\dot{x} = (a_1 - b_1x - c_1y)x \qquad \dot{y} = (a_2 - b_2x - c_2y)y$$

Here a_i, b_i and c_i are positive constants with $c_1a_2 > a_1c_2$ and $b_2a_1 > b_1a_2$. Note that this implies $c_1b_2 > c_2b_1$.

a) Find the equilibria of this system in the closed quarter plane $x \geq 0,\ y \geq 0$.

b) Show that an equilibrium in the open quarter plane $x > 0,\ y > 0$ must be a saddle.

c) Make a plausible phase plane diagram for trajectories in the closed quarter plane.

4. Use the method of characteristics to find a solution to

$$u_t + uu_x = -x, \qquad t \geq 0$$

with $u(x,0) = f(x)$, $-\infty < x < \infty$. You will not be able to find $u(x,t)$ explicitly. However, if $f'(x) \geq 0$, show that the solution will exist for $t \in [0, \pi/2)$.

5. Assume that $y = \phi(x)$ is a smooth, one-to-one mapping of the domain $D \subset \mathbb{R}^2$ onto the domain $\hat{D} \subset \mathbb{R}^2$. Let $\phi'(x)$ be the jacobian matrix of ϕ, and assume that $h(x) = |\det \phi'(x)| \neq 0$. Use the weak form of the equation to show that the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[\beta(x)\frac{\partial u}{\partial x_i}\right] = f \text{ in } D,\ u = 0 \text{ on } \partial D$$

is equivalent to

$$-\frac{1}{\hat{h}(y)}\sum_{i=1}^{2} \frac{\partial}{\partial y_i}\left[\sum_{j=1}^{2} \hat{h}(y)\hat{\beta}_{ij}(y)\frac{\partial \hat{u}}{\partial y_j}\right] = \hat{f} \text{ in } \hat{D}, \hat{u} = 0 \text{ on } \partial\hat{D},$$

where $\hat{u}(\phi(x)) = u(x)$, $\hat{f}(\phi(x)) = f(x)$, $\hat{h}(\phi(x)) = h(x)$, and you need to find the matrix $(\beta_{ij})(y)$.

6. In this problem we have the domains in the (x_1, x_2)-plane

$$\Omega_+^a = \{|x - (1,0)| \le a\} \cap \{x_1 \ge 0\} \text{ and } \Omega_-^a = \{|x - (-1,0)| \le a\} \cap \{x_1 \le 0\},$$

and set $\Omega^a = \Omega_-^a \cup \Omega_+^a$. Consider the Neumann problem

$$\Delta u = f,\ x \in \Omega^a,\ \frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega^a,$$

where $\int_{\Omega_+^a} f dx = 1$ and $\int_{\Omega_-^a} f dx = -1$.

(a) Prove the existence of a solution to this Neumann problem when $a > 1$ and the nonexistence of a solution when $0 < a < 1$.
(b) Show that $\max_{\Omega^a} |\nabla u| \to \infty$ as $a \downarrow 1$. Note that the length of the line segment $L = \Omega_-^a \cap \Omega_+^a$ goes to zero as $a \downarrow 1$.

7. Consider the heat equation, $u_t - \Delta u = 0$, in a bounded domain D in $\mathbb{R}^n$ with the initial condition $u(x, 0) = 0$ and the boundary condition $u(x,t) = f(x)$ on ∂D. Find an expansion for the solution to this problem in terms of eigenfunctions of Δ and the solution of the Dirichlet problem $\Delta w = 0$ in D, $w = f$ on ∂D. What is leading term in the asymptotic expansion of $u(x,t) - w(x)$ as $t \to \infty$?

8. Let $u(x,t)$ be the solution to

$$u_{tt} + a^2(x,t)u_t - \Delta u = 0 \text{ in } D,\ u(x,t) = 0 \text{ on } \partial D$$

with $(u(x,0), u_t(x,0)) = (f(x), g(x))$. Prove that $\int_D u^2(x,t)dx$ is bounded for $t \in [0, \infty)$. You may assume that D is a bounded domain with smooth boundary, f and g are smooth functions vanishing on ∂D, and that a is a smooth function on $D \times [0, \infty)$.