Qualifying Exam on Applied Differential Equations

Tuesday, March 23, 2010, 2PM-6PM.

Solve the following 8 problems. Provide clear and concise arguments. Draw a figure when necessary.

1. Consider the generalized eigenvalue problem

$$y'' - y = -\lambda x^2 y' \text{ for } 0 < x < 1, \quad y(0) = y(1) = 0$$

Show that all eigenvalues λ must be bigger than 1.

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let u be a C^2 solution of the following problem

$$\begin{cases} u_t = \Delta u - u & \text{in} \quad \Omega \times (0, \infty) \\ u(x,0) = g(x) & \text{in} \quad \Omega \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

Suppose $g(x)$ is bounded and compactly supported in Ω.

Using an appropriate energy, show that there exists $C > 0$ such that $|u(x,t)| \leq C \exp^{-t}$ as $t \to \infty$.

3. Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary. Prove the uniqueness of C^2 solution for the following problem:

$$\begin{cases} -\Delta u + a(x)u = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = f(x) & \text{on} \quad \partial\Omega \end{cases}$$

when $a(x) > 0, f(x) \in C^2(\bar{\Omega})$, and ν is the outward normal vector on $\partial\Omega$ with respect to Ω.

4. Let u solve the one-dimensional wave equation

$$u_{tt} - u_{xx} = -u \text{ in } \mathbb{R} \times (0, \infty),$$

with continuous initial data $u(x,0) = g(x), u_t(x,0) = h(x)$ which are both compactly supported.

(a) Find an energy associated with u

(b) Show that $u(\cdot, t)$ is compactly supported at each $t > 0$.

5. Consider the conservation laws $(g(u))_t + (h(u))_x = 0$. Define the notion of integral solution and derive the jump (Rankine-Hugoniot) condition for a discontinuity (u-,u+) in an integral solution.

6. Solve:

$$u_x^2 + yu_y - u = 0, \ (x,y) \in \mathbb{R} \times (1,\infty)$$

$$u(x,1) = \frac{x^2}{4} + 1.$$

7. Let $u : [0,1] \to \mathbb{R}$ be piecewise H^1 with a discontinuity at x_Γ. That is, if $u^- : [0, x_\Gamma) \to \mathbb{R}$ with $u^-(x) = u(x)$ for $0 \le x < x_\Gamma$ and $u^+ : (x_\Gamma, 1] \to \mathbb{R}$ with $u^+(x) = u(x)$ for $x_\Gamma < x \le 1$, then $u^- \in H^1(0, x_\Gamma)$ and $u^+ \in H^1(x_\Gamma, 1)$. Furthermore, define the jump in u at x_Γ as

$$[u] = \lim_{x \to x_\Gamma^+} u(x) - \lim_{x \to x_\Gamma^-} u(x)$$

and $\bar{u}$ as

$$\bar{u} = \frac{1}{2}\left(\lim_{x \to x_\Gamma^+} u(x) + \lim_{x \to x_\Gamma^-} u(x) \right).$$

Show that if

$$\frac{\partial}{\partial x}\left(\beta(x)\frac{\partial u}{\partial x}\right) = 0, \ x \in (0, x_\Gamma) \cup (x_\Gamma, 1),$$

$$[\beta\frac{\partial u}{\partial x}] = \lim_{x \to x_\Gamma^+} \beta(x)\frac{\partial u}{\partial x}(x) - \lim_{x \to x_\Gamma^-} \beta(x)\frac{\partial u}{\partial x}(x) = b,$$

$$u(0) = u(1) = 0 \text{ and } [u] = a$$

(where β is piecewise C^∞ but discontinuous at x_Γ and $\beta(x) \ge \epsilon > 0$) then $e(u) \le e(v)$ for all piecewise H^1 functions v that also satisfy:

$$v(0) = v(1) = 0 \text{ and } [v] = a.$$

Here, $e(u)$ is defined as

$$e(u) = \frac{1}{2}\left[\int_0^{x_\Gamma} \frac{\partial u}{\partial x}(x)\beta(x)\frac{\partial u}{\partial x}(x)dx + \int_{x_\Gamma}^1 \frac{\partial u}{\partial x}(x)\beta(x)\frac{\partial u}{\partial x}(x)dx\right] + \bar{u}b$$

8. Find a solution of the inhomogeneous initial value problem

$$u_t + au_x = f(x,t), \ x \in \mathbb{R}$$

$$u(x,0) = \phi(x).$$