ADE Exam, Fall 2011

Please do All 8 problems.

1. Let a, b be two points in $\mathbb{R}^2$ and consider $U(x) : (\mathbb{R}^2 - \{a,b\}) \to \mathbb{R}$ be a smooth function which satisfies $\limsup_{|q|\to\infty} |U(q)| = 1$. Let us consider a system of ODEs for $(p(t), q(t)) \in \mathbb{R}^2 \times (\mathbb{R}^2 - \{a,b\})$:

$$\begin{cases} \dot{p}(t) = \nabla U(q(t)) \\ \dot{q}(t) = p(t), \end{cases}$$

with initial data $p(0) \in \mathbb{R}^2$ and $q(0) \in \mathbb{R}^2 - \{a,b\}$.

(a) Show that if $0 < T < \infty$ and if $(p(t), q(t))$ is a solution defined on $[0,T)$ then

$$\sup_{t\in[0,T)} |p(t)|, |q(t)| < \infty.$$

(b) Let $[0,T)$ be the maximal interval of existence for $(p(t), q(t))$ with $T < \infty$. Show that $\lim_{t\to T} q(t)$ exists, and moreover

$$\lim_{t\to T} q(t) = a \text{ or } b.$$

2. Let $\vec{v}(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Let $\theta(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a smooth function solving the following equation:

$$\theta_t = \Delta(\theta)^2 + \nabla \cdot (\vec{v}\theta),$$

and $\theta(x,0)$ is bounded from above and below.

(a) Show that θ stays bounded, both from above and below, for all times $t \geq 0$ if $\nabla \cdot \vec{v} = 0$ for all times.

(b) Now suppose that $|\nabla \cdot \vec{v}(x)| \leq M$ for all $x \in \mathbb{R}^n$. If $\theta(x,0) \leq 1$, show that $\theta(x,t) \leq e^{Mt}$ for all $t > 0$.

3. Suppose $u(x,t) : \mathbb{R}^n \to [0,T) \to \mathbb{R}$ is a smooth solution of

$$u_t(x,t) = \Delta u_t + u, \quad u(x,0) = u_0(x) \in L^2(\mathbb{R}^n).$$

with

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2(dx)} = M < \infty.$$

(a) Present $u(x,t)$ in the form of

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^n} G(x-y)u(y,s)dyds$$

for $0 \le t \le T$.

(b) Show that the integral formula in (a) is well-defined in $L^2(dx)$ by showing that

$$\| \int_{\mathbb{R}^n} G(x-y)u(y,s)dyds \|_{L^2(dx)} \le M,$$

for any $0 \le s \le T$.

(c) Using (b), evaluate M in terms of the L^2-norm of $u_0(x)$.

4. Show that all solutions of the nonlinear heat equation

$$u_t = \Delta(u^4) \text{ in } |x| < 1, \quad u = 0 \text{ on } |x| = 1$$

vanishes to zero as $t \to \infty$.

5. Consider the intial value problem

$$u_t + (f(u))_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$u(x,0) = \phi(x).$$

Assume that f is smooth and uniformly convex ($f''(x) > \theta > 0$ $\forall x$ for some $\theta > 0$)

(a) Show that if $\phi(x) = -x$, then there is a point at which $|u_x| \to \infty$ in finite time.

(b) Consider the Riemann initial data:

$$\phi(x) = \begin{cases} u^-, & x < 0 \\ u^+, & x > 0 \end{cases}$$

Compute the entropy solution (show that entropy condition is satisfied). Consider both cases: $u^- > u^+$ and $u^- < u^+$.

6. Consider the initial value problem

$$\begin{aligned} u_{tt} + 2u_{xt} - 3u_{xx} &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

(a) Use energy methods to prove the value of the solution u at the point (x_0, t_0) depends at most on the values of the initial data in the interval $(x_0 - 3t_0, x_0 + t_0)$.

(b) Use energy methods to prove uniqueness of solutions if the initial data has compact support.

7. Let Ω be an open, bounded set in $\mathbb{R}^2$. Let $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \in \Omega$ and let $\mathbf{f} : \Omega \to \mathbb{R}^2$, $\mathbf{f} = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}$ and consider the problem

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}\left(\mathbf{u}\right).$$

(a) Show that all stationary points are saddles if $\nabla \cdot \mathbf{f} = 0$ and $\nabla \times \mathbf{f} = \frac{\partial f_1}{\partial v} - \frac{\partial f_2}{\partial u} = 0$.

(b) Show that $\mathbf{f} : \Omega \to \mathbb{R}^2$ must be C^∞ on Ω.

8. Let Ω be an open set in $\mathbb{R}^n$, and suppose there exists a nonzero function u satisfying

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

for some λ.

(a) Show that λ is positive.

(b) Let λ_1 be the smallest positive such λ. Show that

$$\lambda_1 = \min\{\int_U |Du|^2 dx : u \in H_0^1(U), \int_U u^2(x)dx = 1.\}$$