Applied Differential Equations, Spring 2011.

Do all eight problems. They are each worth 10 points.

1. The equation of motion for a "nonlinear spring" is $y'' = -ky - ay^3$, where $k > 0$ is the constant in Hooke's Law. Rewrite this equation as an equivalent first order system, and analyze the phase plane for it. Indicate the differences between a hard $(a > 0)$ and a soft $(a < 0)$ spring. Also explain what differences you would see if a damping term were added to the equation.

2. Consider the Cauchy problem for equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} - 4\frac{\partial^2 u}{\partial x_3^2} + \frac{\partial u}{\partial x_1} = 0$$

in the half-space $x_1 \cos\theta + x_2 \sin\theta \geq 0$, i.e. the problem of finding a solution to the equation in the half-space with prescribed values and normal derivative on the plane $x_1 \cos\theta + x_2 \sin\theta = 0$.

(a) For what values of θ is this problem characteristic?

(b) For what values of θ is this strictly hyperbolic (with respect to the normal to the plane)?

3. Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary Γ, and assume that $a(x)$ is a continuous function on $\overline{D}$. Show that solutions to $u_t = \Delta u + a(x)u$ vanishing on Γ with $u(0, x) \geq 0$ will be nonnegative for all $t > 0$.

4. Consider the Cauchy problem in the plane

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 1 \qquad u(x, 0) = f(x),$$

where $f \in C^2(\mathbb{R})$. When will this be characteristic at $(x_0, 0)$? Assuming that it is not characteristic at $(x_0, 0)$, find a solution defined in a neighborhood of that point. The solution will be expressed in terms of f and the function $r(x, y)$ defined near $(x_0, 0)$ by $y = (f'(r))^2(x - r - y)$. You should also show that $y = (f'(r))^2(x - r - y)$ has a unique local solution with $r(x_0, 0) = x_0$.

5. (a) Consider the boundary value problem $\Delta u = 0$ in the half-space $x_3 > 0$ in $\mathbb{R}^3$ with $\frac{\partial u}{\partial x_3} = f$ on $x_3 = 0$. Find the kernel $P(x, y)$ such that for f continuous and vanishing for $|x|$ large

$$u(x) = \int_{\mathbb{R}^2} P(x,y) f(y) dy$$

gives a solution to this problem which converges to zero as $|x| \to \infty$.

(b) Show that boundary value problem in (a) can have at most one solution which converges to zero as $|x| \to \infty$.

(c) Suppose that $\int_{\mathbb{R}^2} f(y)dy = 0$. Show that $|u(x)| \leq C|x|^{-2}$.

6. (a) Assume that $a < 1$. Prove that the solution to the mixed problem

$$u_{tt} - \Delta u = 0, \text{ in } \{(t,x) \text{ in } \mathbb{R} \times \mathbb{R}^3,\ x_3 > 0\},\ u_{x_3} - a u_t = 0 \text{ on } x_3 = 0$$

with initial data $u(0, x) = f(x)$, $u_t(0, x) = g(x)$ will vanish in the hemisphere $|x| < R - |t|$, $x_3 > 0$ when f and g vanish for $|x| < R$.

(b) Explain why the result in part (a) is false when $a = 1$.

7. Show that the initial value problem for the nonlinear diffusion equation $u_t = u_{xx} + u^2$ has a solution for short time by completing the following argument. Let $K(t, x)$ be the fundamental solution for the heat equation. You may assume the estimate

$$\sup_{x \in \mathbb{R}} |u(x,t)| \leq \sup_{x \in \mathbb{R}} |u(x,0)|, \quad \text{when } u(x,t) = \int_{\mathbb{R}} K(t, x-y) u(0,y) dy.$$

Given a bounded continuous function $f(x)$, define the sequence of functions $\{u_n\}$ recursively as the solutions to the initial value problems

$$(u_n)_t - \Delta u_n = u_{n-1}^2, \qquad u_n(0,x) = f(x),$$

and show that, for T sufficiently small, this sequence converges uniformly to a solution to $u_t = \Delta u + u^2$ with $u(0, x) = f$ on $\{0 \leq t \leq T\} \times \mathbb{R}_x$.

8. Find a similarity solution $v(t, x) = t^\alpha w(x/t^\beta)$ to $v_t = v_{xx} + (v^2)_x$ First find values for α and β and the differential equation for w which lead to a nontrivial solution v for this equation. Then complete the construction by solving the equation for w.