ADE Exam, Fall 2012

1. Show that the problem

$$\begin{cases} -\Delta u = -1 & \text{for} & |x| < 1, |y| < 1 \\ u = 0 & \text{for} & |x| = 1, \\ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 & \text{for } |y| = 1 & \end{cases}$$

has at most one solution in $|x| > 1, |y| < 1$.

2. Consider the equation

$$\rho_t - \Delta(\rho^2) - \nabla \cdot (2x\rho) = 0, \text{ in } (x,t) \in I\!R^2 \times [0, \infty)$$

where the initial data $\rho_0(x) \geq 0$ is compactly supported and $\int \rho_0 = 1$. Let us assume that $\rho(\cdot, t)$ stays nonnegative and compactly supported for all times $t > 0$. Using formal calulations, show the following.

(a) $\int \rho(\cdot, t) dx = 1$ for all $t > 0$.

(b) The energy

$$\int \rho^2 + \rho |x|^2 + C\rho dx$$

decreases in time for any C.

(c) Using (a) and (b), show that ρ converges to $(C_0 - \frac{|x|^2}{2})_+$ for an appropriate C_0.

3. Consider the ODE

$$u'' + f(u) + \lambda u' = 0$$

for $u \in C^2(\mathbb{R})$, $f \in C^\infty(\mathbb{R})$ and $\lambda > 0$. Prove there are no periodic solutions other than a stationary equilibrium solution ($u \equiv c \in \mathbb{R}$).

4. We say u is a weak solution of the wave equation,

$$\begin{cases} u_{tt} - u_{xx} = 0, -\infty < x < \infty, t > 0. \\ u(x,0) = f(x) \\ u_t(x,0) = g(x), \end{cases}$$

if for all $v \in C_0^\infty(I\!R \times [0, \infty))$ satisfies

$$\int_0^\infty \int_{-\infty}^\infty u[v_{tt} - v_{xx}] dx dt + \int_{-\infty}^\infty f(x) v_t(x,0) dx - \int_{-\infty}^\infty g(x) v(x,0) dx = 0$$

Let $f(x)$ be a piecewise continuous function with a jump at $x = x_0$. Show that $u(x,t) = f(x+t)$ is a weak solution of the wave equation.

5. Consider the following 2×2 systems of ODEs:

(a) $x_t = -y(x^2+y^2)^a; y_t = x(x^2+y^2)^a$

(b) $x_t = -x(x^2+y^2)^a; y_t = -y(x^2+y^2)^a$

For each case, discuss local and global well-posedness of these systems both forward and backward in time as a function of the real parameter a. Also describe the trajectories of the solutions. Recall that well-posedness means existence, uniqueness, and continuous dependence on initial data.

6. Consider the initial value problem

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{u} = \mathbf{0}, \quad t > 0, \quad \mathbf{x} \in I\!R^2$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u_0}(\mathbf{x})$$

where $\mathbf{u} : I\!R^2 \times [0,T) \to I\!R^2$, $T > 0$. That is,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x}u_1 + \frac{\partial u_1}{\partial y}u_2 \\ \frac{\partial u_2}{\partial x}u_1 + \frac{\partial u_2}{\partial y}u_2 \end{pmatrix}.$$

(a) Derive an implicit expression for the solution $\mathbf{u}$ in terms of the initial data $\mathbf{u_0}(\mathbf{x})$.

(b) Give a condition on the eigenvalues of $\frac{\partial \mathbf{u_0}}{\partial \mathbf{x}}$ that will lead to finite time blow up in $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right|$. Hint: recall that $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right| > \rho(\frac{\partial \mathbf{u}}{\partial \mathbf{x}})$ where $\rho(\frac{\partial \mathbf{u}}{\partial \mathbf{x}})$ is the spectral radius.

7. Consider the eigenvalue problem with Neumann boundary conditions

$$-\Delta u = \lambda u, \quad \mathbf{x} \in \Omega$$

$$\nabla u \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega.$$

Let $X_n = \{w \in C^2(\Omega), w \not\equiv 0, \langle w, v_i \rangle = 0, i = 1, 2, \ldots, n-1\}$ where the v_i are the first $n-1$ eigenfunctions. Show that the n^{th} eigenvalue λ_n satisfies

$$\lambda_n = \min_{w \in X_n} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}}.$$

8. Give the entropy satisfying weak solution to Burger's equation

$$u_t + uu_x = 0, \quad u(x,0) = u_0(x)$$

on the periodic domain $[0,4]$ with initial data

$$u_0(x) = \begin{cases} 2, & x \in (0,2) \\ 0, & x \in (2,4) \end{cases} .$$

Show that the slope of the solution is $\frac{1}{t}$ almost everywhere for $t > 2$.