

ADE Exam, Spring 2012

1. Consider the ODE

$$\dot{x} = \sin y \cos x, \quad \dot{y} = -\cos y \sin x.$$

a. Show that the quantity $H(x, y) = \cos x \cos y$ is conserved for trajectories of the above ODE.

b. Compute the fixed points of the above system and identify them as hyperbolic or elliptic fixed points. Sketch a phase portrait of this system.

c. Almost all of the orbits of this system are periodic. Compute the period of these orbits in the following two limits (c1) as the orbit approaches one of the elliptic fixed points and (c2) as the orbit approaches one of the hyperbolic fixed points. More precisely you can consider a point (x_0, y_0) as the initial data for one of the periodic orbits. What is the period of the orbit in the limit as (x_0, y_0) approaches an elliptic fixed point. The same question is asked when the limit is a hyperbolic fixed point.

2. a. Consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t),$$
$$u(x, 0) = \frac{\partial u(x, 0)}{\partial x} = 0,$$

where $x \in \mathbb{R}$, $t > 0$, $f(x, t)$ is smooth, $f(x, t) = 0$ for $t < 0$.

Find an explicit solution of this Cauchy problem.

b. Consider the Cauchy problem

$$(2) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x, t) \frac{\partial u}{\partial t} + b(x, t) \frac{\partial u}{\partial x} + c(x, t) u(x, t) = f(x, t),$$
$$u(x, 0) = \frac{\partial u(x, 0)}{\partial t} = 0.$$

Here a, b, c are smooth functions of (x, t) .

Use part a) to prove existence and uniqueness of a continuously differentiable solution of the Cauchy problem, and prove the estimate

$$\|u\|_{1, \Delta} \leq C \|f\|_{0, \Delta},$$

where Δ is the right triangle with vertices at (x, t) , $(x - t, 0)$, $(x + t, 0)$,

$$\|f\|_{0, \Delta} = \max_{(y, \tau) \in \Delta} |f(y, \tau)|,$$

$$\|u\|_{1, \Delta} = \max_{(y, \tau) \in \Delta} \left(|u(y, \tau)| + \left| \frac{\partial u}{\partial x}(y, \tau) \right| + \left| \frac{\partial u}{\partial t}(y, \tau) \right| \right).$$

Remark. For the simplicity, prove the estimate for $0 < t < \delta$, δ is small, although, the proof will work for any t .

3. Consider traveling wave solutions $U(x-st)$ of the equation $u_t + u^2 u_x = \epsilon u_{xx}$ with left state $u(-\infty) = U_L$ and right state $u(+\infty) = U_R$.

a. Write down the ODE satisfied by U and simplify as much as possible.

b. For what values of $U_L, U_R, \epsilon > 0$ and $s \in \mathbb{R}$ does this ODE have a solution? Justify your answer. Hint: there may be a constraint on the boundary values and the wave speed. If so please derive the formula for this constraint and explain its derivation.

4. Consider the Helmholtz equation in \mathbb{R}^3

$$\Delta u(x) + k^2 u(x) = 0, \quad k > 0.$$

a. Check that $E(x) = \frac{e^{ik|x|}}{4\pi|x|}$ is a fundamental solution.

b. Prove the Green's formula

$$(1) \quad u(x_0) = \int_{\partial B(0,R)} \left(\frac{\partial u}{\partial n} E(x_0 - y) - u \frac{\partial E(x_0 - y)}{\partial n} \right) ds(x),$$

where $x_0 \in \mathbb{R}^3$ is arbitrary, $R > |x_0|$, $B(0, R)$ is the ball of radius R centered at 0, $\frac{\partial}{\partial n}$ is the outward normal to $B(0, R)$.

c. Use the part b to prove the following uniqueness theorem:

Let $(\Delta + k^2)u = 0$ in \mathbb{R}^3 , $u(x) = O(\frac{1}{r})$ and $\frac{\partial u}{\partial r} - iku = o(\frac{1}{r})$, $r = |x|$. Then $u(x) = 0$.

5. Assume $u \in H^1(\Omega)$, $\lambda \in L^2(\partial\Omega)$ satisfy

$$\int_{\Omega} \nabla v \cdot (\beta \nabla u) d\mathbf{x} + \int_{\partial\Omega} \lambda v ds(\mathbf{x}) + \int_{\partial\Omega} \mu u ds(\mathbf{x}) = \int_{\partial\Omega} \mu g ds(\mathbf{x}) + \int_{\Omega} v f d\mathbf{x}, \quad \forall v \in H^1(\Omega) \text{ and } \forall \mu \in L^2(\partial\Omega).$$

Show that this is a variational form of the Dirichlet Poisson problem where we do not have to explicitly assume that $u(\mathbf{x}) = g(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$. However, by omitting this assumption, we must introduce the additional unknown λ . Show that λ is the Neumann boundary condition that would be applied if u were the solution to a Neumann problem. Specifically:

a. Show that u is the solution of the Dirichlet problem:

$$-\nabla \cdot (\beta \nabla u) = f, \quad \forall \mathbf{x} \in \Omega$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega$$

b. Show that if w solves

$$-\nabla \cdot (\beta \nabla w) = f, \quad \forall \mathbf{x} \in \Omega$$

$$\beta(\mathbf{x}) \nabla w(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = -\lambda(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega$$

then $w(\mathbf{x}) = g(\mathbf{x}) + c$ for some constant $c \forall \mathbf{x} \in \partial\Omega$.

6. Consider Burger's equation on the real line

$$u_t + \left(\frac{u^2}{2} \right)_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

with initial data

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty.$$

- a. Derive the classical/strong solution with initial data

$$u_0(x) = x^2$$

This will only be defined for some x and t (specify explicitly which x and t).

- b. Show where the magnitude of the derivative of the strong solution becomes infinite.
c. Consider the entropy satisfying weak solution arising from the piecewise smooth initial data:

$$u_0(x) = \begin{cases} 1/4, & x < -\frac{1}{2} \\ x^2, & -\frac{1}{2} < x < \frac{1}{2} \\ 1/4, & x > \frac{1}{2} \end{cases}$$

Specify when and where the first entropy satisfying shock will appear. Write the ODE that describes the trajectory of the shock (you do not need to solve the ODE).

7. Suppose $u \in C^2(\mathbb{R}^2)$ and

$$\Delta u > 0, \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

Show that $u(\mathbf{x}) \leq \frac{1}{2\pi\epsilon} \int_{\partial B(\mathbf{x}, \epsilon)} u(\mathbf{y}) ds(\mathbf{y})$

8. Let $u(x, y)$ be the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0, \quad k > 0,$$

in the strip $0 < x < \pi$, $-\infty < y < +\infty$, with zero Dirichlet boundary conditions

$$u(0, y) = u(\pi, y) = 0.$$

Find all bounded solutions (if any) of this boundary value problem.