**ADE Exam, Fall 2013**

1. Consider the ODE system

$$x_t = v$$
$$v_t = -\frac{d\psi}{dx}(x) - \alpha v$$

for a given function $\psi(x) \in C^2(\mathbb{R})$.

(a) Find all stationary points and analyze their type when $\psi(x) = \frac{1}{2}(x^2 - 1)^2$.

(b) Sketch the phase plane for $\psi(x) = \frac{1}{2}(x^2 - 1)^2$.

(c) Show that $H(x, v) = \frac{v^2}{2} + \psi(x)$ is non-increasing with time.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$ for $\Omega \subset \mathbb{R}^2$. Let $E(u)$ be defined as

$$E(u) = \int_\Omega f(\frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)) + g(x, y)u(x, y)da + \frac{1}{2}\int_{\partial\Omega} u^2 ds$$

for $u \in C^2(\Omega)$.

(a) Suppose $f(v, w) = \frac{1}{2}\left(v^2 + w^2\right)$ and $g \in L^2(\Omega)$. Show that the minimizer exists in $H^1(\Omega)$.

(b) What differential equation with what boundary equations does the minimizer of $E$ over $u \in C^2(\Omega)$ satisfy? Assume $f \in C^2(\mathbb{R}^2)$ and $g \in L^2(\Omega)$.

3. Consider the following initial value problem

$$\begin{cases} u_t + [f(u)]_x = 0, \ x \in \mathbb{R}, \ t > 0 \\ u(x, 0) = \phi(x) \end{cases}$$

Assume $\exists\theta > 0$ such that $f''(u) \geq \theta \ \forall u$. Show that if $\phi(x) = -x$ then $|u_x| \to \infty$ in finite time.

4. Consider

$$(*) \begin{cases} u_{tt} - c^2 u_{xxxx} + au_t = 0, \quad a > 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

where $\phi$ and $\psi$ have compact support.

(a) For solutions with compact support, define an associated energy $E(t)$ that is non-increasing with $t$.

(b) Use this to show that solutions to $(*)$ with compact support are unique.

5. Let $u(x, t)$ be the solution of the Cauchy problem for the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \ x \in \mathbb{R}, \ t \in (0, T) \\ u(x, 0) = 0 \end{cases}$$

Prove that if $\exists C, a$ such that $\left|\frac{\partial u}{\partial t}(x,t)\right| \le Ce^{ax^2}$, $\left|\frac{\partial u}{\partial x}(x,t)\right| \le Ce^{ax^2}$, $\left|\frac{\partial^2 u}{\partial x^2}(x,t)\right| \le Ce^{ax^2}$ and $|u(x,t)| \le Ce^{ax^2}$ on $t \in (0,T)$ then $u \equiv 0$.

6. Consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial u}{\partial x}(x,t) + u^2(x,t) = 0, \ 0 < t < T \\ u(x,0) = \psi(x) \end{cases}$$

where $\psi$ is smooth with compact support. Prove the existence and uniqueness of smooth solutions when $T$ is small.

7. Let $\Delta u(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^3/\{\mathbf{0}\}$. Suppose $u(\mathbf{x}) = o(\frac{1}{|\mathbf{x}|})$ and $|\nabla u| = o(\frac{1}{|\mathbf{x}|^2})$. Prove that $u$ is harmonic in a neighborhood of the origin too. Hint: Consider the annulus domain: $\epsilon < |\mathbf{x}| < 1$ and find an integral representation of $u(\mathbf{x})$ in this domain.

8. Consider the Neumann problem in the half-plane

$$(*) \begin{cases} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad y > 0, \ x \in \mathbb{R} \\ \frac{\partial u}{\partial y}(x,0) = f(x) \end{cases}$$

(a) Show that if $f(x) = 0$ and $u(x,y) \to 0$ when $x^2 + y^2 \to \infty$ then $u \equiv 0$.

(b) Assume $f(x)$ has a compact support and $\int_{-\infty}^{\infty} f(x)dx = 0$ is zero. Prove that there exists a solution of (*) that tends to zero when $x^2+y^2 \to \infty$ tends to the infinity.