ADE Qualifying Exam, March 2013

Please solve all 8 problems.

1. Consider the Cauchy problem in $\mathbf{R}^n \times [0, +\infty)$:
   (1) $\frac{\partial u}{\partial t} = t^2 \Delta u(x,t), \quad t > 0,\ x \in \mathbf{R}^n,$
   (2) $u(x,0) = g(x), \quad$ where $g(x) \in L_2(\mathbf{R}^n)$.

   Prove that there exists a weak solution $u(x,t)$ of (1), (2) such that $u(x,t)$ is continuous in $t \geq 0$ with values in $L_2(\mathbf{R}^n)$.

   Prove also that
   $$\|u(x,t) - g(x)\|_{L_2(\mathbf{R}^n)} \to 0$$
   when $t \to +0$. (Hint: Use Plancherel's Theorem).

2. Let
   $$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + C(x,t)u = 0$$
   for $t > 0,\ u(x,0) = \varphi(x),\ \frac{\partial u(x,0)}{\partial t} = \psi(x), x \in \mathbf{R}^1$.

   Assume $C(x,t), \varphi(x), \psi(x)$ are smooth functions equal to zero for $|x| > R$. Prove that $u(x,t) = 0$ for $|x| > R + t$.

3. Let D be a domain in $\mathbb{R}^n$ with smooth boundaries.

   a) Let $u(x)$ be a H_1 solution of
   $$-\Delta u + u^{\frac{1}{3}}(x) = 0 \ \text{ in } \ D,$$
   $$u\big|_{\partial D} = 0.$$
   Prove that $u \equiv 0$ in D.

   b) Let $u(x)$ be a $H_1(D)$ solution of
   $$-\Delta u - \alpha u^{\frac{1}{3}} = 0,$$
   $$u\big|_{\partial D} = 0.$$
   Prove that $u \equiv 0$ if $\alpha > 0$ is small. (Hint: Use Poincare's Inequality: $\int_D u^2 dx \leq C \int_D |Du|^2 dx$ for all $u \in H_0^1(D)$.)

4. a) Let

$$\Delta u - q(x)u = 0 \ \text{ in } \ \mathbf{R}^n,$$

where $q(x) \geq 0$ is bounded.
Suppose $u(x) \to 0$ uniformly when $|x| \to \infty$.
Prove that $u(x) \equiv 0$.

b) Find a nontrivial solution of

$$\Delta u + u = 0 \ \text{ in } \ \mathbf{R}^3$$

such that $u(x) \to 0$ when $|x| \to \infty$. (Hint: Consider radial solutions)

5. Consider the following autonomous ODE:

$$y_1' = y_2, \quad y_2' = -y_1 + (1 - y_1^2 - y_2^2)y_2.$$

Show that any solution $x(t) = (x_1(t), x_2(t))$ of above system converges to $(\sin(t+c), \cos(t+c))$ as $t \to \infty$, for some constant c.

6. Draw the phase space for the competing species system

$$x' = x(2 - x - y), \quad y' = y(3 - 2x - y).$$

How likely is it that both species survive?

7. Let Ω be a connected, bounded domain in $\mathbb{R}^n$ with smooth boundary, and let $f(x), g(x) : \mathbb{R}^n \to \mathbb{R}$ be smooth. Show that there is at most one smooth solution of the following equation

$$\begin{cases} u_t - \Delta u + |\nabla u|^2 = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = g(x) & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = f(x) & \text{in} \quad \Omega. \end{cases}$$

8. Show that

$$u(x,t) = -\frac{2}{3}(t + \sqrt{3x + t^2}) \text{ for } 4x + t^2 > 0, \quad u(x,t) = 0 \text{ for otherwise.}$$

is an entropy solution of the equation $u_t + uu_x = 0$.