ADE Exam, Fall 2014

1. An ecosystem contains two species. At time t, there are x individuals of species 1, and y individuals of species 2. The dynamics of the two populations are described by the Lotka-Volterra equations:

$$\frac{dx}{dt} = 2x - x^2 - xy \ , \quad \frac{dy}{dt} = y - xy \tag{1}$$

(a) Describe in words what the terms in the equations might represent.

(b) Sketch the possible trajectories of the ecosystem in the (x, y) phase plane. Your sketch should include any equilibrium points, null-clines and the behavior of trajectories if x and$y$ are both large.

2. (a) Find the eigenfunctions and eigenvalues for $1 \leq x \leq 2$ of the homogeneous ODE:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0 \quad \text{where} \quad y(1) = y(2) = 0 \tag{2}$$

(b) By expanding in these eigenfunctions or otherwise, solve the inhomogeneous ODE:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 3y = x \log x \quad \text{where} \quad y(1) = y(2) = 0 \tag{3}$$

3. Consider Burgers' equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0$$

with initial data $u(x, 0) = \cos(x)$.

(a) Derive an implicit form of the solution $u(x, t)$ in terms of the initial data $u(x, 0)$.

(b) What is $\max\limits_{x \in \mathbb{R}} u(x, t)$? You will need to use an implicit expression in terms of the initial data.

4. Consider the interface problem

$$-\frac{\partial}{\partial x}\left(\beta(x)\frac{\partial u}{\partial x}(x)\right) = f(x), \ x \in (0, 1)/\left\{\hat{x}\right\}, \quad \beta(x) = \begin{cases} 1, & x < \hat{x} \\ 2, & x > \hat{x} \end{cases}$$

(where $\hat{x} \in (0, 1)$) with boundary conditions $u(0) = u(1) = 0$ and interface conditions $u(\hat{x}^+) = u(\hat{x}^-)$ and $\beta(x^+)\frac{\partial u}{\partial x}(\hat{x}^+) = \beta(x^-)\frac{\partial u}{\partial x}(\hat{x}^-)$.

(a) Derive the weak form of the problem.

(b) Solve for the associated Green's function $G(x; x_0)$

$$-\frac{\partial}{\partial x}\left(\beta(x)\frac{\partial G}{\partial x}(x; x_0)\right) = \delta(x - x_0),\ \ x, x_0 \in (0,1)/\{\hat{x}\}$$

with $u(0) = u(1) = 0$ and interface conditions $G(\hat{x}^+; x_0) = G(\hat{x}^-; x_0)$ and $\beta(x^+)\frac{\partial G}{\partial x}(\hat{x}^+; x_0) = \beta(x^-)\frac{\partial G}{\partial x}(\hat{x}^-; x_0)$.

5. Given $\phi \in H^1(0,1)$ with $\phi(0) = 0$, define the energy

$$e(\phi) = \int_0^1 \psi(\frac{\partial \phi}{\partial x}(x))dx - T\phi(1), \quad \psi(F) = \left(F^2 - 1\right)^2.$$

(a) Derive a differential equation (and associated boundary conditions) satisfied by the extrema of this energy.

(b) Are the extrema unique?

6. The *Falkner-Skan* equations describe the boundary layer flow of a fluid over a wedge shaped object. If we set up coordinates so that $y = 0$ corresponds with the plate surface, and y measures distance from the surface, then you may assume the fluid velocity $(u(x,y), v(x,y))$ solves a simplified form of the Navier-Stokes equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (5)$$

with boundary conditions: $u = v = 0$ at $y = 0$ and $u(x,y) \to U(x)$ as $y \to \infty$. Here $U(x)$ represents the velocity of the fluid flow far from the plate, which is given to you. We will assume that $U(x) = U_0 x^m$ for some positive constants U_0 and m. ν is the viscosity of the fluid (a positive constant).

(a) Show that if $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ for some function $\psi(x,y)$, then Eqn. (5) is automatically satisfied. What boundary conditions should be applied on ψ?

(b) What function form should ψ take to reduce Eqn. (4) to an ODE (i.e. can you construct a similarity form for ψ)? Your similarity variable will be of the form $\eta = y/g(x)$ for some function $g(x)$ which you should be able to calculate from scaling analysis of Eqn. (4) and its boundary conditions. You should reduce Eqn (4) to an ODE, but you do not need to solve this ODE.

(c) What boundary conditions need to be introduced for f?

7. Let $U = \{|x| \leq 1\} \subset I\!R^n$. Consider a smooth solution u solving

$$u_t - \Delta u = u^\alpha \text{ in } U_T := U \times (0,T]$$

with boundary data $0 \leq u \leq 1$ on the parabolic boundary of U_T, i.e. on $U \times \{t = 0\}$ and $\partial U \times (0,T]$. Show that if $0 \leq \alpha \leq 1$ then $0 \leq u \leq 2$ in U_T.

8. Let u be a weak solution of the eikonal equation

$$u_t + |u_x|^2 = 0 \text{ in } I\!R \times (0, \infty)$$

with smooth initial data $g(x)$ at $t = 0$. Give an example of g for which u becomes not differentiable in finite time.