ADE Exam, Spring 2014

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1. Consider the initial boundary value problem

$$\begin{cases} (u_t - u_{xx} + u)(x,t) = 0 & \text{for} \quad x > 0, t > 0 \\ u(x,0) = f(x) & \text{for} \quad x > 0 \\ u(0,t) = g(t) & \text{for} \quad t > 0, \end{cases}$$

   where g and f are continuous functions with compact support. Find a solution of this problem.

2. Suppose Ω be a bounded, open set in $\mathbb{R}^n$ with C^2 boundary. Let $u_i(x,t)$, $i = 1,2$ solve the heat equation with the Neumann boundary data

$$\begin{cases} (\partial_t - \Delta)u_i = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \partial_\nu u_i = f & \text{on} \quad \partial\Omega \times (0,\infty). \end{cases}$$

   where $\nu = \nu_x$ denotes the normal vector at $x \in \partial\Omega$, with smooth $f : \partial\Omega \to I\!R$ and smooth initial data.

   (a) Show that

$$A(t) := \sup_{x \in \Omega_1} |u_1 - u_2|(x,t)$$

   decreases in time.

   (b) Suppose u_1 stays uniformly C^2 in space and time. Show that u_1 converges to a constant as $t \to \infty$.

3. Let $v(x,t) = (v_1, ..., v_n)$, $x, t \in I\!R$, solve the $n \times n$ system

$$D_1 v_t + D_2 v_x = 0,$$

   with smooth, compactly initial data $v(x,0)$. Here D_1 and D_2 are constant symmetric matrices, with $D_1 > 0$.

   (a) Let $n = 2$. Use the Fourier transform to find a formula for v in terms of D_1 and D_2 as well as its initial data $v(x,0)$.

   (b) Now answer (a) for $n > 2$. (Hint:one may try to convert the problem into the ODE system of the form $y' = By$.)

4. Let us consider the eikonal equation

$$u_t + \frac{1}{2}|D_x u|^2 = 0 \text{ in } (x,t) \in I\!R^n \times (0,\infty).$$

   with the initial data $u(x,0) = -|x|^2$.

(a) Let $n = 1$, where $D_x u = u_x$. Please explain why we cannot expect a smooth solution.

(b) Answer (a) for $n > 2$.

5. The Euler-Bernouilli equation for beam bending:

$$\rho \frac{\partial^2 w}{\partial t^2} = -EI \frac{\partial^4 w}{\partial x^4} \quad ; \quad 0 < x < L \tag{1}$$

is used to model the deflection $w(x,t)$ of an AFM probe. Here ρ, E and I are all positive constants determined by the shape and material of the probe. We seek the natural modes of vibration of the beam, which are solutions to Eq. (1) of the form $w(x,t) \equiv w(x)e^{i\omega t}$. $w(x)$ must satisfy the eigen-equation:

$$EI \frac{d^4 w}{dx^4} = -\omega^2 w$$

with boundary conditions:

$$w = \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0$$

and:

$$\frac{d^3 w}{dx^3} = \frac{d^4 w}{dx^4} = 0 \quad \text{at} \quad x = L$$

(a) Calculate the Green's function for the eigen-equation.

(b) Show that the eigen-equation is self-adjoint. If $\mathcal{G}$ is the Green's function operator, explain briefly how the quotient $\sup_{u \in C^1[0,L]} \frac{(u,\mathcal{G}[u])}{(u,u)}$ can be used to calculate the lowest normal frequency (smallest value of ω satisfying the eigen-equation).

(c) Using $u(x) = x$ as a test function, use part (b) to estimate the lowest normal frequency.

6. The function $y(t)$ satisfies the ODE:

$$\frac{d^2 y}{dt^2} = -y(1-y)^2 \tag{2}$$

(a) Determine the stability of any stationary points (justify your answers).

(b) Sketch the solution orbits in the (y, y') phase plane.

(c) Now suppose that damping is added to the system:

$$\frac{d^2 y}{dt^2} + |y| \frac{dy}{dt} = -y(1-y)^2 \tag{3}$$

Prove that the point $\left(y, \frac{dy}{dt}\right) = (0,0)$ is asymptotically stable.

7. Let $u \in C^2(\Omega)$ with $u(\mathbf{x}) + \nabla u \cdot \mathbf{n}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$ where $\mathbf{n}(\mathbf{x})$ is the outward normal for $\mathbf{x} \in \partial\Omega$. Consider $r : C^2(\Omega) \to \mathbb{R}$ defined as the scalar $r(u)$ such that

$$E(r(u)) \leq E(\alpha), \quad \forall \alpha \in \mathbb{R}$$

where

$$E(\alpha) = \int_\Omega (\Delta u + \alpha u)^2 \, d\mathbf{x}.$$

(a) Show that

$$r(u) = \frac{\int_\Omega \nabla u \cdot \nabla u d\mathbf{x} + \int_{\partial\Omega} u^2 ds(\mathbf{x})}{\int_\Omega u^2 d\mathbf{x}}.$$

(b) Show that if v minimizes r (over functions that satisfy $v(\mathbf{x}) + \nabla v \cdot \mathbf{n} = 0, \ \mathbf{x} \in \partial\Omega$) that

$$-\Delta v = r(v)v.$$

8. Consider Burgers' equation

$$u_t + f(u)_x = 0$$

for $u(x,t)$ over the periodic domain $x \in (0,1)$, $t > 0$ where $f(u) = \frac{1}{2}u^2$. Solve the periodic initial value problem for $u(x,t)$ with initial data

$$u(x,0) = \begin{cases} 1, & 0 < x < x_0 \\ 0, & x_0 < x < 1 \end{cases}$$

for an arbitrary $x_0 \in (0,1)$.