

ADE Exam, Fall 2015

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the autonomous differential equation system:

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2 \quad (1)$$

- (a) Identify the fixed points of this equation, and show (either by linearizing the equation, or by some other method) whether they are stable or unstable.
- (b) Sketch the trajectories for this differential equation in the (x, y) phase plane. Your sketch should include the eigenvectors of the fixed points identified in part (a) if they correspond to features that can be seen in the phase plane. It should also include the asymptotic behavior of trajectories for large x and/or y .
2. The Lighthill-Whitham-Richards (LWR) models the density of cars, $\rho(x, t)$, on an infinite 1D road with flow in one direction:

$$\partial_t \rho + \partial_x (\rho(1 - \rho)) = 0 \text{ in } \mathbb{R} \times [0, \infty), \quad (2)$$

Here $F = \rho(1 - \rho)$ denotes the flux of cars past a point in the road. The formula for F comes from the assumption that $u(x, t) := 1 - \rho$ is the mean speed of cars on the road.

- (a) Under the assumption that (2) admits a unique solution, show that any state with uniform traffic $\rho(x, 0) = \rho_0$, with $0 < \rho_0 < 1$ is stable. Specifically show that if $\rho(x, t)$ is a solution of (2) for $0 \leq t \leq T$, then $\|\rho(x, t) - \rho_0\|_{L^\infty} \leq \|\rho(x, 0) - \rho_0\|_{L^\infty}$ for all $0 \leq t \leq T$.
- (b) Show that any step discontinuous function of the form:

$$\rho(x, t) = \begin{cases} \rho_\ell & \text{if } x < vt \\ \rho_r & \text{if } x > vt \end{cases} \quad (3)$$

where $0 < \rho_\ell, \rho_r < 1$ are both constants, satisfies the weak form of (2) (which you should derive) so long as the velocity of the discontinuity, v , satisfies a condition, which you should also derive.

- (c) The weak solutions for this PDE are not unique. To impose uniqueness, it is often assumed that car accelerations must be bounded. Namely, if $x = X(t)$ is the trajectory of a car on this road:

$$\frac{dX}{dt} = u(X(t), t) \quad (4)$$

then $\frac{d^2X}{dt^2} < \infty$ (cars can decelerate infinitely quickly, so $\frac{d^2X}{dt^2}$ can be arbitrarily large and negative). Show that under this assumption solutions of the form given in part (b) are allowed only if $\rho_\ell < \rho_r$.

3. Consider the porous medium equation with drift:

$$\rho_t - \Delta(\rho^3) - \nabla \cdot (2x\rho) = 0, \text{ in } (x, t) \in \mathbb{R}^2 \times [0, t],$$

where the initial data $\rho_0(x) \geq 0$ is compactly supported and $\int \rho_0(x) dx = 1$. Let us assume that $\rho(\cdot, t)$ stays nonnegative and compactly supported for all times $t > 0$. Using formal calculations, show the following.

- (a) $\int \rho(\cdot, t) dx = 1$ for all $t > 0$.
 (b) Show that the energy

$$E(t) := \int \left[\frac{1}{2} \rho^3(x, t) + \rho |x|^2 + C\rho \right] dx$$

decreases for all times $t > 0$ for any constant C . (Hint: Show that $\frac{d}{dt} E(t) = - \int (|\nabla F(\rho, x)|^2 \rho) dx$ for some function $F(\rho, x)$.)

- (c) Using (a) and the function $F(\rho, x)$ derived from (b), show that ρ converges as $t \rightarrow \infty$ to the stationary profile

$$\left(\max(0, A - B|x|^2/2) \right)^{1/2}$$

for appropriate A and B .

4. A 1D free boundary problem models the solidification of a super-cooled liquid. The function $x = a(t)$ represents the boundary between the solid and liquid, and needs to be solved for. At time t liquid occupies the domain $x > a(t)$ (solid phase occupies $0 \leq x < a(t)$). In the liquid phase, the temperature is given by the heat equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{for } x > a(t) \quad \text{with } T(\infty, t) = 0 \quad (5)$$

The following boundary condition is used at $x = a(t)$:

$$T(a(t), t) = 1 \quad \text{and} \quad S \frac{da}{dt} = - \left. \frac{\partial T}{\partial x} \right|_{x=a(t)^+} \quad (6)$$

where S is a dimensionless number. Show that provided $S > 1$, equation (5) taken together with boundary condition (6) admits a similarity solution. You will need to solve both for the temperature field $T(x, t)$ and for the free boundary location $a(t)$. Show that your solution breaks down if $S < 1$.

Hint 1: To construct your similarity solution you may find it useful to make use of the complementary error function defined as $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$.

Hint 2: Additionally, you may assume that if a function $F(z) = e^{z^2} z \operatorname{erfc}(z)$ is defined for $z \geq 0$, then $F(z)$ is monotonic increasing.].

5. Let $U = \{|x| \leq 1\} \subset \mathbb{R}^n$. For a given time $T > 0$, consider a smooth solution u solving the PDE:

$$u_t - \Delta u = u(u - 1) \text{ in } U_T \equiv U \times (0, T]$$

with boundary data $0 \leq u < 1$ on the parabolic boundary of U_T , i.e. on $U \times \{t = 0\}$ and $\partial U \times (0, T]$. With these assumptions, prove that $0 \leq u < 1$ in the entire domain U_T . You should show the proof of maximum principle if you use it.

6. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz, and let u be the unique weak solution of the Hamilton-Jacobi equation

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g(x) & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where D is the operator: $D \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$.

(a) Show that there exists some $C > 0$ such that $|u(x, t) - g(x)| \leq Ct$ for any $t > 0$.

(b) Suppose that $|g(x)| \leq M|x|^{-1}$ with M a constant. Show that $u(x, t)$ converges to zero as $t \rightarrow \infty$.

7. Consider K be the set of functions $u : [0, 2] \rightarrow \mathbb{R}$ of the form

$$u(x) = \begin{cases} v(x) & \text{for } x \in [0, 1) \\ w(x) & \text{for } x \in (1, 2] \end{cases}$$

and $v \in C^2[0, 1)$, $w \in C^2(1, 2]$, with the property that $v(0) = w(2) = 0$ and $[u] := w(1) - v(1) = a$. You should assume that the value of the constant a is known. Define

$$E(u) = \frac{1}{2} \int_0^1 (u_x)^2 dx + \int_1^2 (u_x)^2 dx + \bar{u}b$$

where $\bar{u} := \frac{w(1)+v(1)}{2}$, and b is another constant, whose value you can assume is known. Show that there exists $h(x)$ that minimizes E over all functions $u \in K$, and solve for $h(x)$.

8. Let $u(x, t)$ be the entropy solution of the Burgers' equation

$$u_t + f(u)_x = 0, x \in \mathbb{R}, t > 0$$

with $f(u) = \frac{u^2}{2}$ and initial data

$$u(x, 0) = \begin{cases} (x+1)^2 & \text{for } x \in [-1, 0] \\ (x-1)^2 & \text{for } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Find the time $T > 0$ when $u(\cdot, t)$ become discontinuous for the first time.