ADE Exam, Spring 2015

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the damped conservation law

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = -u(x,t),\ x \in \mathbb{R},\ t \in [0,\infty)$$

$$u(x,0) = u^0(x),\ x \in \mathbb{R}$$

where $u^0(x)$ has compact support. Define the notion of integral solution and derive the jump (Rankine-Hugoniot) condition for a discontinuity (u^-, u^+) in an integral solution.

2. Show that there is at most one solution to

$$u_{tt} - (c(x))^2 u_{xx} + u_t = 0,\ x \in \mathbb{R},\ t \in [0,\infty)$$

$$u(x,0) = \phi(x),\ x \in \mathbb{R}$$

$$u_t(x,0) = \psi(x),\ x \in \mathbb{R}$$

where ϕ and ψ are smooth and $\exists \hat{c} > 0$ with $|c(x)| \leq \hat{c}$.

3. Solve for $\mathbf{u} : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^2$ that satisfies

$$\frac{\partial u_i}{\partial t}(\mathbf{x},t) + \sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j}(\mathbf{x},t)u_j(\mathbf{x},t) = -u_i(\mathbf{x},t),\ \mathbf{x} \in \mathbb{R}^2,\ t \in [0,\infty),\ i = 1,2$$

with initial conditions

$$\mathbf{u}(\mathbf{x},0) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

4. Consider the energy

$$E(u) := \frac{1}{8}\int_0^1 |u'(x)|dx + \int_0^1 |u(x) - g(x)|^2 dx,$$

where $g(x) = 0$ if $0 < x < 1/2$, $g(x) = 1$ if $1/2 < x < 1$.

(a) Find the piecewise constant function

$$u(x) = \begin{cases} u_l, & x < \frac{1}{2} \\ u_r, & x > \frac{1}{2} \end{cases}$$

with minimal energy. Here, for non C^1 functions u, the total variation $\int_0^1 |u'(x)|dx$ is defined in the weak sense, i.e.,

$$\int_0^1 |u'(x)|dx := \sup\{\int_0^1 u(x)\phi'(x)\, dx, \text{ where } \phi \in C_0^1([0,1]) \text{ with } |\phi(x)| \leq 1\}.$$

(b) Show that there is no C^1 function $u(x)$ with energy lower than the optimal piecewise constant function in part (a).

5. Consider

$$u_t - \Delta u + u^2 = 0 \text{ in } \mathbb{R}^n \times [0,T].$$

Suppose u and v are bounded solutions of above problem with $|u|, |v| \leq M$, $|u|, |v| \to 0$ as $|x| \to \infty$ and

$$|u(\mathbf{x},0) - v(\mathbf{x},0)| < \epsilon.$$

then show that we have

$$|u(\mathbf{x},t) - v(\mathbf{x},t)| < \epsilon \exp^{2Mt} \text{ for all } t > 0.$$

Hint: note that $u^2 - v^2 \leq |u+v||u-v| \leq 2M|u-v|$.

6. The function $y(x)$ satisfies the boundary value problem:

$$\frac{d^2y}{dx^2} - \frac{6y}{x^2} = f(x)$$

with boundary conditions $y = 0$ at $x = 0$ and $y \to 0$ as $x \to \infty$.

(a) Find the Green's function solution of this equation (i.e. solve the boundary value problem in the case where $f(x;\tilde{x}) = \delta(x - \tilde{x})$).

(b) Now suppose that $f(x) = 1$ for $x < 1$ and $f(x) = 0$ for $x > 1$, (i) What are the appropriate continuity conditions on y at $x = 1$? (ii) Use the Green's function from part (a) to solve the ODE.

7. A droplet spreads under gravity on a horizontal surface. The height of the droplet is given by a function $h(x,t)$.

It can be shown that the height of a droplet $h(x,t)$ evolves according to the following porous medium equation:

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right),$$

with $(-L(t), L(t)) = \{x : h(x,t) > 0\}$. In other words $x = \pm L(t)$ is the *free boundary* associated with h. $L(t)$ increases as the droplet spreads – you will have to solve for $L(t)$ as well as for $h(x,t)$.

Note that due to the divergence form of the PDE its volume is conserved, i.e., if we set the initial volume of the drop to be 1, then for all times $t > 0$ we have

$$\int h(x,t)dx = 1.$$

Find a similarity solution for $h(x,t)$.

Hint #1: Assume that $h(x,t) = a(t)h(x/\ell(t))$ for functions $a(t)$ and $\ell(t)$ that you will need to determine.
Hint #2: You may find it useful to make use of the identity:

$$\int_0^{\pi/2} \cos^{5/3}\theta \, d\theta = \frac{3\sqrt{\pi}\Gamma(4/3)}{5\Gamma(5/6)} \approx 0.84$$

8. (a) Sketch the phase plane for the following dynamical system:

$$\dot{x} = x(1-y^2) \qquad \dot{y} = y(1-x^2)$$

Your sketch should include behavior of trajectories that start near the equilibrium points [it is sufficient to determine what the type of each equilibrium point is; you do not need to calculate the eigenvectors], and of any trajectories that connect equilibrium points, along with the asymptotic form of the trajectories for large x and y.

(b) Suppose instead that the dynamical system was modified slightly to read:

$$\dot{x} = x(1-y^2) \qquad \dot{y} = y^2(1-x^2)$$

Prove that the equilibrium point $(x,y) = (1,-1)$ is stable.