ADE Exam, Fall 2016

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Show that $u(x) = -\frac{1}{4\pi} r$, where $x \in \mathbb{R}^3$ with $r = |x|$, satisfies $\Delta u = \delta(x)$ in the sense of distributions, i.e.,

$$\int u(x) \Delta \phi(x) dx = \phi(0)$$

for any smooth ϕ with compact support.

2. Consider the following nonlinear drift-diffusion equation

$$\theta_t = \Delta(\theta^2) + \nabla \cdot (x\theta) \quad \text{ for } (x,t) \in \mathbb{R}^n \times (0, \infty).$$

(a) Let $\theta_1(x,t)$ and $\theta_2(x,t)$ be two smooth, nonnegative solutions of the above equation with ordered initial data $\theta_1(x,0) \leq \theta_2(x,0)$. Show that then $\theta_1(x,t) \leq \theta_2(x,t)$ for all $t > 0$.

(b) Show that for any constant $C > 0$, the function $U(x) := \max[(C - \frac{1}{4}x^2), 0]$ is a weak stationary solution of above equation, i.e.,

$$\int_{\mathbb{R}^n} (\nabla(U^2) + xU) \cdot \nabla \phi(x) dx = 0$$

for any compactly supported, smooth $\phi : \mathbb{R}^n \to \mathbb{R}$.

3. Consider the system of ODE for the pair $(x(t), y(t))$ of real-valued functions,

$$\begin{cases} \frac{\partial x}{\partial t} = v \\ \frac{\partial v}{\partial t} = -\frac{1}{2} x(x^2 - 1) - v \end{cases}$$

for $t > 0$, with initial conditions $x(0) = x_0$ and $v(0) = v_0$.

(a) Find all stationary points and sketch the local trajectories.

(b) Define a non-zero function $E(a, b)$ such that $E \geq 0$ and $\frac{d}{dt} E(x(t), v(t)) \leq 0$ for all $t > 0$ when $(x(t), v(t))$ solves the ODE system.

4. Consider $\phi(x,t) : I\!R^n \times (0,\infty) \to I\!R$ solving the Hamilton-Jacobi equation

$$\phi_t + |\nabla\phi| = 0,$$

with initial data $\phi(x,0) = \max(|x|^2 - 1, 0)$. Show that $\phi(x,t) = 0$ when $t = |x| - 1$.

5. Consider the smooth solution u of the Dirichlet problem

$$\begin{cases} -\nabla \cdot (\beta(x)\nabla u(x)) = 0 & \text{in} \quad \Omega, \\ u(x) = g(x) & \text{in} \quad \partial\Omega. \end{cases}$$

where Ω is a smooth, bounded domain in $I\!R^n$ and the functions $\beta : \Omega \to I\!R^+$, $g : \partial\Omega \to I\!R$ are smooth. Suppose there exists a bijective map $\boldsymbol{\phi} : \Omega \to \hat{\Omega}$ satisfying

$$D\boldsymbol{\phi}(x) = \frac{1}{\beta(x)} Q(x),$$

where Q satisfies $\det(Q) = 1$ and $Q^T Q = I$. Show that then $u(x) := \hat{u}(\boldsymbol{\phi}(x))$ is the smooth solution of

$$\begin{cases} -\Delta \hat{u}(\hat{x}) = 0, & \text{in} \quad \hat{\Omega}, \\ \hat{u}(\hat{x}) = g(\boldsymbol{\phi}^{-1}(\hat{x})) & \text{in} \quad \partial\hat{\Omega}. \end{cases}$$

6. Let D be a subset of $\mathbb{R}^n$. Show that the smallest constant C for the Poincaré Inequality

$$\int_D u^2 dx \le C \int_D |\nabla u|^2 dx \text{ for all } u \in H_0^1(D)$$

can be obtained from an eigenvalue problem. State the eigenvalue problem and explain why.

7. Let us consider the one dimensional reaction diffusion equation

$$u_t - u_{xx} = f(u) \text{ in } (x,t) \in \mathbb{R} \times (0,\infty),$$

where $f(0) = f(1) = 0$, $f'(0) > 0$, and $0 < f(u) < f'(0)u$ for $0 < u < 1$. We would like to show that there is a positive *traveling wave* solution with speed c, i.e. solution of the form

$$u(x,t) := U(x - ct) > 0 \text{ with } u(-\infty) = 1, u(+\infty) = 0,$$

for the range of the speeds $c \ge 2\sqrt{f'(0)}$.

To show our claim, first observe that U and $V := U'$ satisfies the two dimensional ODE system

$$U' = V, \quad V' = -(cV + f(U)).$$

We will analyze the (U,V) phase plane to find the traveling wave solution, which connects the steady states $(0,0)$ and $(1,0)$ of the system.

(a) By studying the ODE system near $(0,0)$, conclude that there are no positive traveling wave solution if $c < 2\sqrt{f'(0)}$. By studying the ODE system near $(1,0)$, show that there is at most one traveling wave solution.

(b) For $c \geq 2\sqrt{f'(0)}$, let λ be one of the eigenvalues at $(0,0)$. Show that $V'/U' < \lambda$ on the line $V = \lambda U$.

(c) Using (b) and other observations from phase plane, show that there is exactly one trajectory connecting $(1,0)$ to $(0,0)$ in the (U,V)-system when $c \geq 2\sqrt{f'(0)}$.

8. Let u be the solution of the wave equation in three dimensions

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases}$$

where the initial data is supported in the ball of radius R about the origin. Let x_0 be a point in $\mathbb{R}^3$ with $|x_0| > R$.

a. Find the largest time T_1 for which we can guarantee that $u(x_0,t)$ must be zero for all $0 \leq t < T_1$.

b. Find the smallest time T_2 for which we can guarantee that $u(x_0,t)$ must be zero for all $t > T_2$.