ADE Exam, Spring 2016

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1.   (a) Show that the point $(x, y) = (-1, 0)$ is a stable fixed point for the system of differential equations:

$$\dot{x} = 4y^3 \ , \quad \dot{y} = -2(x + 1)$$

   (b) Now consider the following modification of the system of differential equations:

$$\dot{x} = 4y^3 + (x + 1) - (x + 1)((x + 1)^2 + y^4) \ , \quad \dot{y} = -2(x + 1) + 2y^3 - 2y^3((x + 1)^2 + y^4)$$

Show that the modified system has a limit cycle.

2. Consider the conservation law

$$g(u)_t + f(u)_x = 0, \quad x \in \mathbb{R}, \ t > 0$$

for functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$.

   (a) Derive the Rankine-Hugoniot condition for shock speeds.

   (b) Use the result from (a) to solve the following Riemann problem with $g(u) = \frac{u^2}{2}$, $f(u) = \frac{u^3}{3}$ and with initial data

$$u(x, 0) = \begin{cases} 0, & x \leq 0 \\ 1, & x \in (0, \frac{1}{3}) \\ 0, & x \geq \frac{1}{3} \end{cases}$$

3. In this question you will construct a Green's function solution to Poisson's equation in the domain, $[0, 1] \times \mathbb{R}^2$, that is, solve:

$$\nabla^2 u = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

in the domain $0 \leq x \leq 1$ and $-\infty < y, \ z < \infty$. You may assume that $u \to 0$ as $|y| \to \pm\infty$ or $|z| \to \pm\infty$, and the boundary conditions on the walls $x = 0$, $x = 1$ are:

$$u(0, y, z) = \frac{\partial u}{\partial x}\Big|_{(1, y, z)} = 0$$

Seek a solution of the form:

$$u(x, y, z) = \frac{1}{(2\pi)^2} \iint e^{i(\ell y + mz)} \hat{u}(x, \ell, m) \, d\ell \, dm \tag{1}$$

and find the function: $\hat{u}$. You do not need to evaluate the integral (1).

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4. Solve the Hamilton-Jacobi equation

$$\phi_t + |\phi_x| = 0, \ x \in \mathbb{R}, \ t > 0$$

with initial data

$$\phi(x,0) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

5. A toy model for the propagation of an action potential along a neuron is given by the PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)$$

where $f(u)$ may be assumed to be continuously differentiable. Propagating action-potential solutions of this PDE are given by traveling waves, i.e. solutions of the form $u(x,t) = u(x - ct)$, that tend to (different) constant values: $u \to u_*^-$ as $x \to -\infty$ and $u \to u_*^+$ as $x \to +\infty$.

   (a) Explain why the limits of $x \to \pm\infty$ must correspond to values u_* at which $f(u_*) = 0$ and $f'(u_*) \leq 0$.

   (b) Suppose $u_*^- < u_*^+$ and $u(x - ct)$ is monotone increasing in $\eta = x - ct$. Prove that the wave moves leftward (that is $c < 0$) or rightward according to whether:

$$\int_{u_*^-}^{u_*^+} f(u)\, du \gtrless 0$$

   (c) Now consider the specific function $f(u) = -(u - u_0)(u - u_1)(u - u_2)$ where $u_0 < u_1 < u_2$ are all constants. Guessing that $\frac{du}{d\eta} = B(u - u_0)(u - u_2)$ for some constant B, find the traveling wave solution, $u(\eta)$ and its velocity c.

6. Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$, centered at the origin, and let u be a smooth solution of

$$\begin{cases} u_{tt} + a^2(x)u_t - \Delta u = 0 & \text{in} & B_1(0) \times (0, \infty), \\ u(x,t) = 0 & \text{on} & \partial B_1(0) \times (0, \infty) \\ u(x,0) = g(x), u_t(x,0) = h(x) & \text{on} & B_1(0) \times \{0\}. \end{cases}$$

Here g, h and a are smooth functions and g, h vanishes on $\partial B_1(0)$. Prove that $\int_{B_1(0)} u^2(x,t)dx \leq C\exp(-At)$, where $A = \min\{a^2(x)\}$ and C only depends on g, h and n.

7. Consider the system of PDEs:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}u + \rho\frac{\partial u}{\partial x} = 0, \ x \in \mathbb{R}, \ t > 0$$

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u\right) = k\frac{\partial}{\partial x}\left(\frac{\rho^0}{\rho}\right), \ x \in \mathbb{R}, \ t > 0$$

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for $u(x,t)$ and $\rho(x,t)$, where $\rho^0(x) = \rho(x,0)$ and $k > 0$. Also consider $\phi : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ defined by $\frac{\partial \phi}{\partial t}(X,t) = u(\phi(X,t),t)$ and $\phi(X,0) = X \ \forall X \in \mathbb{R}$ where u and ρ are solutions of the above.

(a) Define $R(X,t) = \rho(\phi(X,t),t)$ and $J(X,t) = \frac{\partial \phi}{\partial X}(X,t)$. Show that $R(X,t)J(X,t) = R(X,0)$.

(b) Show that $R(X,0)\frac{\partial^2 \phi}{\partial t^2}(X,t) = k\frac{\partial^2 \phi}{\partial X^2}(X,t)$, $X \in \mathbb{R}$ $t > 0$.

(c) Use the results in (a) and (b) so solve the system with initial conditions $u(x,0) = c \sin(x)$ and $\rho(x,0) = 1$ where $|c| < \sqrt{k}$. You can express your answer in terms of the inverse (f^{-1}) of the function $f(X) = X + \hat{c} \sin(X)$ (which exists provided that $|\hat{c}| < 1$).

8. Consider the parabolic PDE

$$\begin{cases} u_t - \Delta u + |u_{x_1}| = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x) & \text{in} \quad \mathbb{R}^n, \end{cases}$$

where g is a continuous function with compact support. Show that there is at most one solution of the above problem that tends to zero as $|x| \to \infty$.