Fall 2017, ADE qualifying exam

September 13th, 2017

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

Problem 1. Consider the differential equation:

$$\ddot{x} + x^n \dot{x} + x = 0$$

where n is a non-negative integer.
(a) If n is even, show that the equilibrium $(x, \dot{x}) = (0, 0)$ is asymptotically stable.
(b) If $n = 1$, what can you say about the stability of $(x, \dot{x}) = (0, 0)$?

Problem 2. A chemical diffuses freely in 1D, satisfying the following partial differential equation:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \delta(x)\Theta(t) \tag{1}$$

(here, $\Theta(t)$ is the Heaviside function, which satisfies $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ otherwise). The term on the right hand side represents a point source of the chemical, which introduces chemical at a constant rate, switching on at $t = 0$.

Construct a similarity solution of this partial differential equation for $c(x, t)$. You may assume that $c(x, 0) = 0$ for $x \neq 0$, and that $\lim_{x\to\pm\infty} c(x, t) = 0$.

Problem 3. Consider the initial value problem:

$$y'' + \frac{yy'}{x^4} + y^2 = 0 \quad , \quad y(0) = y'(0) = 0$$

Determine whether or not there exists a unique solution of this differential equation in a neighborhood of $x = 0$.

Problem 4. Solve for entropy satisfying weak solutions of Burgers' equation

$$u_t + uu_x = 0, \; x \in \mathbb{R}, \; t > 0$$

with initial data

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 1 - x, & 0 \le x < 1 \\ 0, & 1 < x \end{cases}$$

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Problem 5. (a) Solve for the Green's function $G(\cdot;\hat{x}) : [0,1] \to [0,1]$ with

$$-\frac{\partial^2 G}{\partial x^2}(x;\hat{x}) = \delta(x - \hat{x})$$

for $\hat{x} \in (0,1)$ and $G(0;\hat{x}) = G(1;\hat{x}) = 0$.

(b) Define $a(w,v) = \int_0^1 w_x(x)v_x(x)dx$ and $(v,f) = \int_0^1 v(x)f(x)dx$ for $f \in L^2(0,1)$. Let $u \in H^1(0,1)$ with $u(0) = u(1) = 0$ and

$$a(u,v) = (v,f), \forall v \in H^1(0,1)$$

with $v(0) = v(1) = 0$. Similarly define $u^h \in H^1(0,1)$ with $u^h(0) = u^h(1) = 0$ and

$$a(u^h, v^h) = (v^h, f), \forall v^h \in W^h$$

where $W^h = \left\{ v^h \in H^1(0,1) | \exists v_i \in \mathbb{R} \text{ such that } v^h(x) = \sum_{i=1}^N v_i N_i(x) \right\}$. Here $h = \frac{1}{N+1}$, $x_i = ih$ and

$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

Show that $G(\cdot;x_i) \in W^h$ and use this to show that $u^h(x_i) = u(x_i)$.

Problem 6. Consider the wave equation

$$\begin{aligned} u_{tt} &= c^2 \nabla u, \ x \in \mathbb{R}^3, \ t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

where the initial data f and g are only non-zero in the region $a < ||x|| < b$. (You should interpret $||x||$ as the 1-norm of x: $||x|| = |x_1| + |x_2| + |x_3|$.). Given a point x, find the time $T > 0$ such that $u(x,t) = 0$ for $0 < t < T$ when
(a) $||x|| > b$
(b) $||x|| < a$.

Problem 7. For a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, consider a smooth solution u of the parabolic PDE

$$\begin{cases} u_t - \Delta u = (M - u)_+ & \text{in} \quad \Omega \times (0,\infty); \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty); \\ u(x,0) = g(x) & \text{in} \quad \Omega, \end{cases}$$

where f_+ denotes $\max(f, 0)$ and g is a smooth function which vanishes on $\partial\Omega$.

Show that if $g(x) \le M$ then $u(x,t) \le M$ for all $t > 0$. (Hint: compare u with $M + \varepsilon t$.) You may apply the Hopf's lemma for the heat equation without proving it, but please explain how you use it.

Problem 8. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Suppose there exists a minimizer u of the functional

$$E(u) = \int_\Omega \frac{1}{2}|Du|^2(x)dx,$$

among smooth functions w in $\bar{\Omega}$, with the constraints

$$w = 0 \text{ on } \partial\Omega, \quad \int_\Omega w^2(x)dx = 1.$$

(a) Show that, for any smooth function w in $\bar{\Omega}$, there exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that $w(\tau) := \tau w + \phi(\tau)u$ satisfies $\int_\Omega (u + w(\tau))^2 = 1$ for sufficiently small $\tau > 0$ (the range of τ depends on the choice of w).

(b) Show that $\phi'(0) = -\int_U uwdx$.

(c) One can use (a) and (b) to perturb the energy to derive the boundary value problem that u satisfies. Find the PDE problem, which involves the constant $\lambda := \int_\Omega |Du|^2 dx$.