ADE Exam, Spring 2017

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Let us consider the continuity equation $\rho_t + \nabla \cdot (\rho \vec{v}) = 0$ in $I\!R^3 \times (0, \infty) \to I\!R$ with $\vec{v}(x) : I\!R^n \to I\!R^n$ and initial data ρ_0.

   (a) Represent ρ in terms of ρ_0 using the method of characteristics, assuming that $\vec{v}$ is Lipschitz continuous. Explain where the Lipschitz continuity assumption is used in the argument.

   (b) Suppose $-1 < \nabla \cdot \vec{v}$ in $I\!R^3$ and $\rho_0 = \chi_{|x|<1}$, where χ_A denotes the characteristic function of a set A. Show that then $\Omega_1 := \{x : \rho(x, 1) > 0\}$ has its volume greater than 4/3.
   (Hint: you may use the fact, which follows from your answer for (a), that the solution of $u_t + \nabla u \cdot \vec{v} = 0$ shares the same characteristic path as ρ).

2. Consider the following parabolic equation

$$\theta_t = \Delta((|x|^2 + 1)\theta) + |D\theta| - 4n\theta \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0, \infty).$$

   (a) Let $\theta_1(x,t)$ and $\theta_2(x,t)$ be two smooth, nonnegative solutions of the above equation which vanishes at inifinity, with ordered initial data $\theta_1(x,0) \le \theta_2(x,0)$. Show that then $\theta_1(x,t) \le \theta_2(x,t)$ for all $t > 0$.

   (b) Let θ be a smooth, nonnegative, integrable solution of above equation, where all its derivatives and its products with $|x|^2$ vanish at $|x| \to \infty$. Show that $\int \theta(\cdot, t)dx$ exponentially decays to zero as $t \to \infty$.

3. Let u solve the following boundary value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \{(t,x) \in I\!R^+ \times I\!R^3, x_1 > t/2\}; \\ u_t = 4u_{x_1} & \text{on} \quad \{x_1 = t/2\}. \end{cases}$$

   Show that $u(x,t) = 0$ in $\{|x| < R - t\} \cap \{x_1 > t/2\}$ when $u(x,0) = u_t(x,0) = 0$ in $\{|x| < R\} \cap \{x_1 > 0\}$. Explain where the boundary condition on $\{x_1 = t/2\}$ has been used.

4. Let $\mathcal{V}^k = \text{span}\{q_1, q_2, \ldots, q_k\}$, $q_i \neq 0 \in L^2(0,1)$, $q_i(0) = q_i(1) = 0$, $\int_0^1 q_i(x) q_j(x) dx = \delta_{ij}$ where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Define $\mathbf{A} \in \mathbb{R}^{k \times k}$ with $a_{ij} =$

$\int_0^1 \frac{\partial q_i}{\partial x}(x)\frac{\partial q_j}{\partial x}(x)dx$ with eigenvalue decomposition $\mathbf{A} = \mathbf{V}\hat{\mathbf{\Lambda}}\mathbf{V}^T$ where $\hat{\mathbf{\Lambda}}$ has diagonal entries $\hat{\lambda}_i$ and $\mathbf{V}$ is orthogonal with entries v_{ij}, $i, j = 1, 2, \dots, k$. Show that $r_i \in \mathcal{V}^{k\perp} = \left\{ f \in L^2(0,1) \,|\, \int_0^1 f(x)q(x)dx = 0 \;\forall q \in \mathcal{V}^k \right\}$ with $r_i(x) = -\frac{\partial^2 w_i}{\partial x^2}(x) - \hat{\lambda}_i w_i(x)$ and $w_i(x) = \sum_j v_{ij} q_j(x)$.

5. Consider the PDE

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\;\; x \in (0,1),\;\; t > 0$$

$$u(x,0) = (s-1)x,\;\; \frac{\partial u}{\partial t}(x,0) = 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0,\;\; \frac{\partial u}{\partial x}(1,t) = 0.$$

for constant $s \in \mathbb{R}$.

(a) Solve the PDE. Hint: solve in terms of the even extension ($u^e : \mathbb{R} \to \mathbb{R}$) of the initial data where

$$u^e(x) = \begin{cases} (s-1)\hat{x}, \;\; \hat{x} \in [0,1) \\ (s-1)(2-\hat{x}), \;\; \hat{x} \in [1,2) \end{cases}$$

with $\hat{x} = 2(x/2 - \text{floor}(x/2))$ for $x \in \mathbb{R}$. floor(y) is the closest integer to y with floor$(y) \leq y$.

(b) Define $e(t) = \int_0^1 \left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx$. Show that $e(t) = (s-1)^2$.

6. Explain whether the ordinary differential equation:

$$5y'' + \left(\frac{y'}{x}\right)^2 + 4y^2 = 0$$

has a unique smooth solution in a neighborhood of $x = 0$, when initial conditions $y(0) = 1$, $y'(0) = 0$ are applied.

Hint: Start by making the change of variables $y(x) = 1 + cx^2 + v(x)$, where $v(x) = o(x^2)$ as $x \to 0$, and c is a constant that needs to be determined.

7. Evolutionary rock-paper-scissors games are used to model interactions among bacteria. Consider three species of bacteria, with relative abundances R (rock), P (paper) and S (scissors) respectively. You may assume that $P+R+S = 1$. A R-type bacteria tends to out compete S-type bacteria, but is itself outcompeted by P-type bacteria.

The growth rate of the R-population is therefore proportional to the number of interactions each R-type bacteria has with S-types, minus the number of interactions with P types. That is:

$$\dot{R} = R(S - P)$$

similarly:

$$\begin{aligned} \dot{S} &= S(P - R) \\ \dot{P} &= P(R - S) \end{aligned}$$

(a) Describe all of the possible behaviors of the system if $R = 0$ at $t = 0$.
(b) Show that, if all three population types are present in the system at $t = 0$ (i.e. R, P, S are all initially non-zero), then none of the types of bacteria will go extinct. That is, none of the variables P, R or S converges to 0, as $t \to \infty$.

8. The space $y > 0$ is filled with a non-Newtonian fluid, initially at rest. A plate at $y = 0$ is set into motion at time $t = 0$. The fluid velocity, $u(t, y)$ then obeys an equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial \tau}{\partial y} \ , \ t > 0, \ y > 0, \tag{1}$$

with boundary conditions:

$$u(t, 0) = 1 \ , \quad \text{and } u(t, +\infty) = 0 \tag{2}$$

and initial condition $u(0, y) = 0$ for $y > 0$. τ is assumed to obeys a constitutive equation:

$$\tau = \left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

(a) Try to derive a similarity solution i.e. look for a solution of the form $u(t, y) = f(\eta)$ where $\eta = y/\delta(t)$ for some function $\delta(t)$, that you will need to determine), by applying only the boundary condition $u(t, 0) = 1$. Show that this similarity solution can not be compatible with the other boundary condition, or with the initial condition.

(b) To find a solution that is compatible with all boundary and initial conditions we modify the constitutive equation to:

$$\tau = \begin{cases} \left(\frac{\partial u}{\partial y}\right)^2 & \text{if } \frac{\partial u}{\partial y} < 0 \\ 0 & \text{if } \frac{\partial u}{\partial y} \geq 0 \end{cases} \tag{4}$$

Derive a similarity solution that satisfies all of the initial and boundary conditions.

Hint: Start by assuming that the solution breaks down into two parts: $0 < y < Y(t)$, in which $\tau \neq 0$ and $y > Y(t)$ in which $\tau = 0$. Derive continuity conditions that must be applied at $y = Y(t)$. You need to solve for the function $Y(t)$, as well as for $f(\eta)$.