ADE Exam, Fall 2018
Department of Mathematics, UCLA

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. (a) Consider the dynamical system

$$\begin{aligned} \dot{x} &= ax + by\,, \\ \dot{y} &= cx + dy\,, \end{aligned} \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ (with $ad - bc \geq 0$) are constants. Classify the equilibrium at $(0, 0)$ for all possible choices of the four constants. Indicate clearly all bifurcations that occur, and designate when you get closed orbits.

(b) Consider the dynamical system

$$\begin{aligned} \dot{x} &= -y + \alpha x(x^2 + y^2)\,, \\ \dot{y} &= x + \alpha y(x^2 + y^2)\,, \end{aligned} \tag{2}$$

where $\alpha \in \mathbb{R}$ is a constant. Determine, with appropriate arguments, the stability of the equilibrium point at the origin. Also draw the phase portraits for this system for all qualitatively different values of α.

2. Consider the equation

$$xy''(x) + (2x - 1)y'(x) + \frac{1}{x}y(x) = 0\,, \tag{3}$$

where prime denotes differentiation with respect to x.

(a) Classify, with a mathematical argument, the points $x = 0$ and $x = \infty$ as ordinary, regular, or irregular singular points.

(b) For $x = 0$, determine the indicial equation and indicial exponents. Find the series expansion about $x = 0$ for the solution of (3) that satisfies

$$y'(0) = 1\,,$$

and from it obtain the solution in closed form. Why is one initial condition sufficient to determine this solution uniquely?

3. Consider the Chebyshev equation

$$\frac{d}{dx}\left((1 - x^2)^{1/2}\frac{dy}{dx}\right) + n^2(1 - x^2)^{-1/2}y(x) = 0\,, \quad -1 < x < 1\,, \tag{4}$$

for integers $n \geq 0$.

(a) Find the general solution to (4)

(b) Denote by $T_n(x) = \cos(n \arccos(x))$ the degree-n polynomial solution of (4). Show that the $T_n(x)$ satisfy the orthogonality relation

$$\int_{-1}^{1} T_n(x)T_m(x)(1 - x^2)^{-1/2}\, dx = 0\,, \quad \text{for } m \neq n\,.$$

Determine the expansion of the function $g(x) = (1 - x^2)^{1/2}$ in terms of the $T_n(x)$.

4. For a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, consider the parabolic PDE

$$\begin{cases} u_t - \Delta u = (1-u)_+ & \text{in} \quad \Omega \times (0,\infty)\,; \\ u(x,t) = l(x) & \text{on} \quad \partial\Omega \times (0,\infty)\,; \\ u(x,0) = g(x) & \text{in} \quad \Omega\,, \end{cases}$$

where g is a smooth function that vanishes on $\partial\Omega$.

(a) Show that if $l(x), g(x) \le 1$, then $u(x,t) \le 1$ for all $t > 0$.

(b) Show that if $l(x), g(x) > 1$, then $u(x,t) > 1$ for all $t > 0$.

5. Consider the following initial–boundary-value problem for $u = u(x,t)$ in the domain $\{x > 0\} \times \{t > 0\}$:

$$\begin{cases} u_t - u_{xx} + au = 0 & \text{in} \quad \{x>0\} \times \{t>0\}\,; \\ u(x,0) = f(x) & \text{on} \quad \{x>0\}\,; \\ u(0,t) = g(t) & \text{on} \quad \{t>0\}\,, \end{cases}$$

where $f(x)$ and $g(t)$ are continuous functions with compact support and a is a constant. Find an explicit solution of this problem.

6. For a bounded domain Ω in $\mathbb{R}^n$ and for

$$u \in \mathcal{A} := \{u \in C^1(\Omega) \text{ with } u = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega u = 1\}\,,$$

consider the energy

$$E(u) := \int_\Omega \sqrt{1 + |Du|^2}\, dx\,.$$

(a) Show that $E(u)$ has at most one minimizer among $u \in \mathcal{A}$.

(b) Let $\Omega = \{|x| < 1\}$ and suppose that u^* minimizes $E(u)$ in $\mathcal{A}$. Show that u is a radial function.

7. (a) Consider the linear equation

$$u_t + au_x = 0\,, \tag{5}$$

with $a > 0$. Solve the initial–boundary-value problem for equation (5) in the domain $x > 0$, $t > 0$ with boundary conditions $u(x,0) = 0$, $u(0,t) = 1$. Draw a characteristic diagram for this problem (a graph of the solution in the x–t plane).

(b) Consider the nonlinear equation

$$u_t + (u^3)_x = 0 \tag{6}$$

for viscous flow down an inclined plane. Solve the initial–boundary-value problem for equation (6) in the domain $x > 0$, $t > 0$ with boundary conditions $u(x,0) = 0$, $u(0,t) = 1$. Here $x = 0$ corresponds to a a gate that releases fluid with height $u(0,t)$. Draw a characteristic diagram for this problem.

(c) Consider the same problem in (b), but now with the boundary conditions $u(x,0) = 1$, $u(0,t) = 0$, corresponding to a uniform flow with the gate closing at time 0. Find a solution that is continuous in the domain $x \ge 0$, $t > 0$. Draw a characteristic diagram for this problem. Is the solution uniformly continuous? Explain your answer.

8. The equation of motion of a vibrating beam is

$$-c_m u_{tt} = EIu_{xxxx} \, ,$$

where u is the displacement of the beam as a function of position along its axis, the constant $c_m = \rho A$ is the linear mass density of the beam, E is the elastic modulus, and I is the moment of inertia.

If the beam is simply supported at the ends, it satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ (no displacement at the ends) and $u_{xx}(0,t) = u_{xx}(L,t) = 0$ (zero bending moments).

(a) Compute the solution of this problem, given the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$.

(b) Find the solution of the vibrating-string equation

$$u_{tt} = c^2 u_{xx} \, ,$$

with fixed boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Compare how the spectrum of the normal modes scales with the length of the string versus the length of the beam.