ADE Exam, Spring 2018

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the following non-dimensionalized model for glycolysis:

$$\dot{x} = -x + ay + x^2 y\,,$$
$$\dot{y} = b - ay - x^2 y\,, \tag{1}$$

where $x \geq 0$ is the concentration of ADP, $y \geq 0$ is the concentration of F6P, and $a, b > 0$ are kinetic parameters. Determine the equilibrium points and their linear stability, and show that a periodic orbit exists if and only if a and b satisfy an appropriate condition (which you should determine). Draw the phase portrait in this case.

2. Consider the ordinary differential equation

$$Ly \equiv -(p(x)y')' + q(x)y = \delta(x - \xi)\,, \tag{2}$$

where $x \in (a, b)$ and the function $p(x) \neq 0$ on (a, b).

   (a) Derive the Green's function solution $G(x, \xi)$. Your answer will contain the unknown functions $p(x)$ and $q(x)$.

   (b) Give an alternative expression for $G(x, \xi)$, in terms of an eigenfunction expansion, and show that the two formulas agree.

3. Consider the ordinary differentiation equation

$$x^3 \frac{d^2 y}{dx^2} + y = 0\,. \tag{3}$$

   (a) Show that the ODE has a regular singular point at $x = \infty$ and determine its indicial exponents.

   (b) The leading behavior of a particular solution to (3) is $t(x) \sim x$ (as $x \to \infty$). By considering the largest terms in a singular series solution, determine the next-largest term in the expansion of $y(x)$ for large positive x.

4. We seek a solution of $u\ : \Omega \times \mathbb{R} \to \mathbb{R}$ solves the partial differential equation:

$$u_t = \Delta u - u||Du||$$

where $\Omega \subset \mathbb{R}^d$ is the interior of a connected compact set, and $||\cdot||$ is the usual Euclidean norm. If smooth boundary conditions ($u(x, t) = f(x, t)$ on $\partial\Omega$) and initial conditions ($u(x, 0) = g(x)$) are specified, show that there is at most one $C^{1,2}(\Omega \times \mathbb{R})$ solution of this PDE.

5. Consider entropy solutions, $u(x,t) \; : \; \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ of the flux-conservation equation:

$$u_t + \big(f(u)\big)_x = 0$$

with initial condition

$$u(x,0) = \begin{cases} x, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

and flux function $f(u) = \frac{u^3}{3}$.

(a) Derive the Rankine-Hugoniot condition for the propagation of discontinuous solutions of this PDE.

(b) Find the long time solution of the PDE. You may assume that $u \geq 0$, so $f(u)$ is convex, and also that at long times, the solution can be broken into three parts:

$$u(x,t) = \begin{cases} 0, & \text{if } x < 0, \\ t^\alpha g\left(\dfrac{x}{t^\beta}\right), & \text{if } 0 < x < h(t), \\ 0, & \text{if } x > h(t). \end{cases}$$

for some exponents α and β, and positive functions g and h, all of which you should determine.

6. *Elastodynamics* is the study of how waves of vibration propagate through elastic bodies. Small deformations of an infinite elastic body can be modeled by a displacement vector field $\boldsymbol{u}(\boldsymbol{x},t) \; : \; \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. This displacement field must satisfy an equation:

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla\cdot\boldsymbol{u}) - \mu\nabla\times(\nabla\times\boldsymbol{u})$$

where ρ (density) and λ, μ (elastic moduli) are all positive constants.

(a) Suppose that we define fields $\omega = \nabla\times\boldsymbol{u}$ and $\epsilon = \nabla\cdot\boldsymbol{u}$. Show that these fields both satisfy wave equations and find the corresponding wave speeds in terms of the constants ρ, λ and μ.

(b) Assume that the displacement field within the solid is spherically symmetric; that is: $\boldsymbol{u}(\boldsymbol{x},t) = u(r,t)\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is a unit vector in the radial direction, and $r^2 = x^2 + y^2 + z^2$. In this case, show that $\omega = 0$.

(c) Assume a spherically symmetric displacement field, with initial conditions $\epsilon(r,0) = \dfrac{\phi(r)}{r}$ and $\epsilon_t(r,0) = 0$. (You may assume that ϕ is C^2, with $\phi(0) = 0$)

(i) Derive an expression for $\epsilon(r,t)$.

(ii) Given the initial condition:

$$\epsilon(r,0) = \begin{cases} 1, & \text{if } r < 1, \\ 0, & \text{if } r > 1. \end{cases}$$

calculate $u(r,t)$ (you can assume that $\lim_{r\to\infty} u = 0$). Hint: To relate u and ϵ, it is helpful to calculate $\int_{B(0,r)} \epsilon \, dV$.

7. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of smooth boundary $\partial\Omega$. Recall the notation $g \in C^1(\bar{\Omega})$ means there exists an open set O containing $\bar{\Omega}$ such that $g \in C^1(O)$.

   Let $f_1, \cdots, f_d \in C^1(\bar{\Omega})$ be such that

$$\sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega.$$

   Suppose $u \in C^2(\bar{\Omega})$ and

$$\triangle u + \sum_{i=1}^{d} f_i \frac{\partial u}{\partial x_i} - u^3 - u^5 = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

   Show that u is identically zero on Ω.

8. Let $\Phi \in C^3(\mathbb{R}^d)$ be such that Φ and its first derivatives are bounded. We consider the Lagrangian

$$L(x, v) = \frac{1}{2}|v|^2 - \Phi(x).$$

   Given $0 < T \leq \infty$ and and $x, y \in \mathbb{R}^d$, we define the minimal action

$$C(x, y) = \inf_{\sigma} \left\{ \int_0^T L(\sigma(\tau), \dot{\sigma}(\tau)) d\tau \;\middle|\; \sigma \in C^1\big([0, T]\big)\,,\; \sigma : [0, T] \to \mathbb{R}^d,\; \sigma(0) = x, \sigma(T) = y \right\}. \tag{4}$$

   (a) Show that if $\Phi \equiv 0$ then $\sigma_0(\tau) = \big(1 - \frac{\tau}{T}\big)x + \frac{\tau}{T}y$ is the unique path minimizing (4).

   (b) Show that if Φ is concave then (4) has at most one minimizer.