

ADE Exam, Fall 2019
Department of Mathematics, UCLA

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the Lorenz system

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= rx - xz - y, \\ \frac{dz}{dt} &= xy - bz,\end{aligned}\tag{1}$$

where $x, y, z \in \mathbb{R}$ and the parameters are $\sigma, r, b > 0$.

- (a) Show that equation (1) has an equilibrium point at the origin and find all other equilibrium points.
(b) Using linear stability analysis, classify the equilibrium point at the origin and find a polynomial equation that is satisfied by the eigenvalues of any other equilibrium point.

For the latter equilibrium point(s), find a pair of pure imaginary eigenvalues when $r = r_H$ (where you need to express $r_H = r_H(\sigma, b)$ in terms of σ and b) with suitable values of σ and b (which you need to specify). In this situation, also find the third eigenvalue.

2. Obtain the Green's function for each of the following boundary-value problems, and use it to express the solution for the given data.

- (a)

$$Ly \equiv y''(x) - \mu^2 y'(x) = f(x), \quad -\infty < x < 0,$$

with boundary conditions

$$y \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty, \quad y(0) = 1.$$

Note that $\mu \neq 0$ is a constant.

- (b)

$$Ly \equiv y''(x) - (1 + x^2)y(x) = f(x), \quad -\infty < x < \infty,$$

with the regularity boundary conditions that the solution vanishes at $\pm\infty$.

3. (a) Find the general homogeneous solution of the Cauchy–Euler equation

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + (1 + \alpha)y = 0, \quad (2)$$

where α is a given positive constant.

- (b) Use (a) to determine the eigenvalues and eigenfunctions of the self-adjoint problem

$$\frac{d}{dx} \left(x^3 \frac{dy}{dx} \right) + \lambda xy = 0, \quad y(1) = 0, \quad y(e) = 0. \quad (3)$$

- (c) Obtain the eigenfunction expansion for the solution of the nonhomogeneous problem

$$\frac{d}{dx} \left(x^3 \frac{dy}{dx} \right) = x, \quad y(1) = 0, \quad y(e) = 0. \quad (4)$$

Give the coefficients explicitly (i.e., compute the integrals).

4. Let u^ϵ , with $\epsilon > 0$, be the entropy-satisfying weak solution of

$$\begin{aligned} u_t^\epsilon + u^\epsilon u_x^\epsilon &= 0, \\ u^\epsilon(x, 0) &= \begin{cases} \frac{x+\epsilon}{\epsilon^2}, & x \in [-\epsilon, 0] \\ \frac{\epsilon-x}{\epsilon^2}, & x \in [0, \epsilon] \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Find $\lim_{\epsilon \rightarrow 0} u^\epsilon(x, t)$.

5. Consider functions $v, f : [a, b] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ that satisfy

$$\begin{aligned} \rho \frac{\partial v}{\partial t} &= \frac{\partial p}{\partial x}, \quad x \in (a, b), \quad t > 0 \\ p &= k(f - 1), \\ \frac{\partial f}{\partial t} &= \frac{\partial v}{\partial x} + \frac{1}{W}(1 - f), \quad x \in (a, b), \quad t > 0, \end{aligned} \quad (5)$$

with boundary conditions $p(a, t) = p(b, t) = 0$ and initial conditions $v(x, 0) = 0$, $f(x, 0) = s$, with $s \in (0, 1)$.

- (a) Show that $\frac{\partial e}{\partial t} \leq 0$, with

$$e(t) = \int_a^b \rho \frac{v^2}{2} + \frac{k}{2}(f - 1)^2 dx,$$

for positive constants ρ , k , and W .

- (b) Solve the system (5) in terms of the even extension of the initial data $v^0(x)$ when $\frac{1}{W} = 0$.

6. Consider the heat equation $u_t = u_{xx}$, with $x \in \mathbb{R}$ and initial condition $u(x, 0) = H(x)$, where H denotes the Heaviside function (which is equal to 0 for $x < 0$ and to 1 for $x > 0$).

- (a) Prove that the solution to this problem has infinite speed of propagation. That is, show for any $y < 0$ and any time $t > 0$, that $u(y, t) > 0$.

[Hint: Consider the formula for the solution using the heat kernel.]

- (b) Consider the same initial condition, but now with the evolution equation

$$u_t = -u_{xxxx}.$$

Prove that the solution to this problem also has infinite speed of propagation.

[Hint: Consider the kernel $k_4(x, t)$ whose Fourier transform satisfies

$$\widehat{k_4(x, t)} = \exp(-16\pi^4 \xi^4 t).$$

You may assume the following properties of k_4 : it decays at infinity, it is symmetric, and it is self-similar (specifically, $k_4(x, t) = K_4(x/t^{1/4})$). You may use without proof the fact that K_4 changes sign and that it decays faster than any polynomial.]

7. Consider the viscous Burgers equation

$$u_t + uu_x = \epsilon u_{xx}, \quad x \in \mathbb{R},$$

with far-field boundary conditions $u(-\infty, t) = U_L$ and $u(+\infty, t) = U_R$.

- (a) Derive the equation for a similarity solution in the form of a traveling wave $u(x, t) = U(x - st)$, and solve for s in terms of U_L and U_R .
- (b) Write a necessary and sufficient condition that U_L and U_R must satisfy for a traveling-wave solution to exist.
- (c) Show that the similarity solution has an additional scaling parameter ϵ . Assuming that ϵ is a length scale, write the similarity solution in dimensionless variables with $\tilde{x} = x/\epsilon$.
- (d) Compute the limiting solution as $\epsilon \rightarrow 0$.

8. Consider the Navier–Stokes equations in two space dimensions:

$$\begin{aligned} v_t + v \cdot \nabla v &= -\nabla p + \nu \Delta v, \\ \nabla \cdot v &= 0, \end{aligned}$$

where $v : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a velocity field, p is a scalar function, $\nu > 0$ is a constant, and Δ denotes the Laplacian. Let $\omega(x, t) = -\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}$ denote the scalar vorticity.

- (a) Show that ω satisfies the equation

$$\omega_t + v \cdot \nabla \omega = \nu \Delta \omega.$$

- (b) Because $\nabla \cdot v = 0$, there exists a scalar function $\Psi(x, t)$ such that $v = \left(-\frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial x_1}\right)^T$. Verify that any v of this form satisfies the divergence-free condition.
- (c) Derive an equation for ω in terms of Ψ . Show that its solution is

$$\Psi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y| \omega(y, t) dy.$$

- (d) Derive an equation for v in terms of ω .