Spring 2019, ADE Qual

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Let $u(x,t)$ solve the initial value problem

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

(a) Derive a formula for u in terms of g and h, when g and h are C^2.

(b) Next consider the boundary value problem

$$\begin{cases} u_{tt} + u_{xt} - 2u_{xx} = 0, & 0 \le x \le 1, t > 0, \\ u(0,t) = u(1,t) = 0; \\ u_t(0,t) = u_t(1,t) = 0. \end{cases}$$

Show that a smooth solution u to above problem must be zero if $u(x,0) = 0$.

2. Consider $\rho(x,t)$ solving the nonlinear diffusion equation with drift:

$$\rho_t - \Delta(\rho^2) - \nabla \cdot (x|x|^2\rho) = 0 \quad \text{in } (x,t) \in \mathbb{R}^2 \times (0,\infty),$$

where the initial data $\rho(x,0) = \rho_0(x) \ge 0$ is compactly supported and satisfies $\int \rho_0(x)dx = 1$. Let us assume that $\rho(\cdot,t)$ stays nonnegative and compactly supported for all times $t > 0$. Using formal calculations, show the following.

(a) Show that $\int \rho(\cdot,t)dx \equiv 1$ for all $t > 0$.

(b) Show that the energy

$$E(t) := \int_{\mathbb{R}^2} [\rho^2(x,t) + \rho(x,t)\frac{|x|^4}{4} + C\rho(x,t)]dx$$

non-increases in time for any constant C.

(c) Using (a)-(b) show that $\rho(\cdot,t)$ converges as $t \to \infty$ to the stationary profile $(C_0 - |x|^4/4)_+$ for an appropriate constant C_0.

3. Let $u \in C^2(\{|x| < 1\}) \cap C^1(\{|x| \le 1\})$ solve the elliptic equation

$$\begin{cases} -\Delta u = 1 - u^2 & \text{in} \quad \{|x| < 1\}, \\ \nabla u \cdot x = f(x) & \text{on} \quad \{|x| = 1\}. \end{cases}$$

Show that $u \le 1$ if $f \le 0$.

4. Let u be the entropy satisfying weak solution of

$$u_t + f(u)_x = 0,\ x \in \mathbb{R},\ t > 0$$
$$u(x,0) = \begin{cases} u_a, & x < 0 \\ u_b, & x > 0 \end{cases}$$

with $f(u) = \frac{u^2}{2}$ with $u_a, u_b > 0$.

(a) Show that

$$\int_a^b u(x,t)dx - \int_a^b u(x,0)dx = t\left(f(u_a) - f(u_b)\right)$$

for $t < T$.

(b) Give an expression for T with $u_a < u_b$.

(c) Give an expression for T with $u_a > u_b$.

5. Consider the bi-harmonic boundary value problem

$$u_{xxxx} = f, x \in (0,1)$$
$$u_x(0) = u_x(1) = u(0) = u(1) = 0.$$

(a) Show that the Green's function $G(\cdot;\hat{x}) : [0,1] \to [0,1]$ satisfies

$$\int_0^1 v_{xx}(x) G_{xx}(x;\hat{x})dx = v(\hat{x}).$$

for v that satisfy $v_x(0) = v_x(1) = v(0) = v(1) = 0$.

(b) Show that the Green's function satisfies

$$G(x;\hat{x}) = \begin{cases} \sum_{i=0}^{3} \frac{a_i(x-\hat{x})^i}{i!}, & x < \hat{x} \\ \sum_{i=0}^{3} \frac{b_i(x-\hat{x})^i}{i!}, & x > \hat{x} \end{cases}$$

and derive a linear system for the coefficients a_i and b_i.

6. A heated plate lies along the positive x-axis, and air flows over the plate in the x-direction. There is a simple shear flow above the plate. Assume that the plate surface is at a temperature T_0. Thus for $x > 0$, $y > 0$, the temperature field above the plate is given by:

$$\gamma y \frac{\partial T}{\partial x} = D \frac{\partial^2 T}{\partial y^2}$$

Here γ (the shear rate), D (the diffusivity) and T_0 are all positive constants. Assume that at $x = 0$ the air is at ambient temperature $T(0, y) = 0$, while on the surface of the plate $T(x, 0) = T_0$, and far from the plate: $T(x, \infty)) = 0$. Construct a similarity solution of the PDE of the form: $T(x, y) = a(x) f\left(\frac{y}{L(x)}\right)$. You should determine the functions: $a(x)$, $L(x)$ and $f(\cdot)$.

7. Consider the ordinary differential equation

$$y'' + \frac{yy'}{x^4} + y'^2 = 0 \ , \ y(0) = 1 \ , \ y'(0) = 0$$

By considering all possible dominant balances as $x \to 0$, determine whether or not the equation admits a unique solution near $x = 0$.

8. Let u be a solution of Poisson's equation in a domain U:

$$-\nabla^2 u = f(x)$$

for some smooth function $f(x)$. The piecewise smooth boundary of U can be divided into two subsets with measure 0 intersection, we call these sets ∂U_N and ∂U_D. On ∂U_N, the normal derivative $\frac{\partial u}{\partial n} = N(x)$ is known, whereas on ∂U_D, $u(x) = h(x)$ is prescribed.

(a) Show that $u(x)$ minimizes the functional:

$$E[u] = \int_U \left(\frac{1}{2}||\nabla u||^2 - fu\right) dV - \int_{\partial U_N} Nu \, dS$$

among a set of C^2 functions that you should identify.

(b) Suppose that we are studying flow in a pipe whose cross-section is the unit square $|x|, |y| < 1$. The flow field solves the Poisson equation:

$$-\nabla^2 u = 1$$

On three of the walls of the pipe: $x = -1$ and $y = \pm 1$, we have $u = 0$. On the fourth wall ($x = 1$) $\frac{\partial u}{\partial x} = 0$. Explain how you could use the minimization principle from part (a) to calculate optimal constants A, $x_{1,2}$, $y_{1,2}$, in an approximation to the flow field of the form:

$$u = A(x_1 - x)(x_2 - x)(y_1 - y)(y_2 - y) \ .$$

(You do not need to evaluate your integrals to calculate A explicitly, but you should identify the constants $x_{1,2}$ and $y_{1,2}$).