ADE Exam, Fall 2020
Department of Mathematics, UCLA

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. (a) [7 points] Show that the dynamical system

$$\begin{aligned} \dot{x}_1 &= -2x_2 + x_2x_3\,, \\ \dot{x}_2 &= x_1 - x_1x_3\,, \\ \dot{x}_3 &= x_1x_2 \end{aligned} \tag{1}$$

has a stable equilibrium point at the origin.

(b) [3 points] By suitably modifying the dynamical system (1), construct a dynamical system for which the origin is asymptotically stable but whose linearization does not give a sink at the origin.

2. Find two linearly independent solutions of the equation

$$2t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0\,, \quad 0 < t < \infty\,. \tag{2}$$

3. Consider a smooth solution that satisfies the elliptic equation

$$-\Delta u = (u+1)(u-1) \text{ in } \{|x| < 1\}\,, \quad u = f(x) \text{ on } |x| = 1\,,$$

where f is a continuous function such that $-1 < f(x) < 1$ on $|x| = 1$. Prove carefully that $-1 < u \leq 1$ in $|x| \leq 1$.

[*Note: It is not sufficient to simply cite the maximum principle without reasoning.*]

4. Consider the following divergence-form elliptic equation with smooth coefficients:

$$Lu = -\nabla \cdot (A(x)\nabla u) + c(x)u \quad \text{in} \quad U := |x| < 1\,,$$

where A is a positive-definite and symmetric matrix, with the Neumann boundary condition $x \cdot (A(x)\nabla u) = 0$ on $|x| = 1$.

Let (w_k, λ_k), with $\|w_k\|_{L^2(U)} = 1$, be the kth eigenfunction and eigenvalue associated with L with the Neumann boundary condition.

(a) [2 points] Show that $\{w_k\}$ is an orthonormal basis of $L^2(U)$.

(b) [3 points] Show that the smallest eigenvalue of L is

$$\lambda_1 = \min_{u \in H^1(U), u \neq 0} E(u) := \frac{\int_U [(A(x)\nabla u) \cdot \nabla u + c(x)u^2] dx}{\int_U u^2}\,.$$

(c) [5 points] Show that $\lambda_1 < 0$ if $\int_U c(x)dx < 0$.

5. Let ρ be a smooth solution of

$$\rho_t - \Delta\rho - \nabla\rho \cdot x - n\rho = 0 \quad \text{in} \quad (x,t) \in \mathbb{R}^n \times [0,\infty)\,,$$

where the initial data $\rho_0(x) \geq 0$ is compactly supported and $\int \rho_0 = 1$. Assume that $\rho(\cdot, t)$ is positive with exponential decay as $|x|$ tends to infinity for all $t > 0$.

(a) [2 points] Show that $\int \rho(\cdot, t)dx = 1$ for all $t > 0$.

(b) [3 points] Show that the energy

$$\int \left(\rho(x) \ln \rho(x) + \rho(x) \frac{x^2}{2} \right) dx$$

decreases in time.

(c) [5 points] What are the possible limit profiles of $\rho(x,t)$ as t tends to infinity? Please explain.

6. Consider the initial/boundary-value problem for the inviscid Burgers equation $u_t + uu_x = 0$ on the quadrant $x > 0$, $t > 0$, with boundary conditions $u(x,0) = 0$ and $u(0,t) = 1$ for $0 < t < 1$ and $u(0,t) = 0$ for $t \geq 1$.

(a) [5 points] Draw a characteristic diagram for the solution for the time interval $0 < t < 2$. Assume that all shocks satisfy the entropy condition. Write down the entropy condition.

(b) [3 points] Determine the solution and its total mass $M = \int_0^\infty u(x,2)dx$ at $t = 2$. Show that this mass is conserved for $t > 2$.

(c) [2 points] Determine the solution for $t > 2$.

7. (a) [2 points] Show that the general solution of the PDE $u_{\xi\eta} = 0$ is $u(\xi, \eta) = F(\xi) + G(\eta)$.
   (b) [2 points] Using the change of variables $\xi = x + t$ and $\eta = x - t$, show that $u_{xx} - u_{tt} = 0$ if and only if $u_{\xi\eta} = 0$.
   (c) [6 points] Consider the wave equation $u_{xx} - u_{tt} = 0$ on the real line with initial conditions $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$. Using the notation above, find a formula (up to a constant) for F and G in terms of g and h such that $u = F(x + t) + G(x - t)$ satisfies the above initial value problem.

8. Consider the Cahn–Hilliard equation on the 2-torus T^2 (i.e., periodic boundary conditions in two dimensions):

$$u_t = -\Delta \left(\epsilon \Delta u - \frac{1}{\epsilon} W'(u) \right) , \tag{3}$$

where W is the double-well potential $(u^2 - 1)^2$, the operator Δ is the Laplacian in two dimensions, and $\epsilon > 0$ is a small parameter.

   (a) [2 points] Show that smooth solutions of equation (3) conserve mass: $\int u = C$.
   (b) [2 points] Show that smooth solutions of equation (3) cause the Ginzburg–Landau energy

$$\int_{T^2} \left[\frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right] dx$$

   to be nonincreasing.
   (c) [6 points] Show that the Ginzburg–Landau energy is non-convex, with two locally stable equilibria ($u(x, t) = 1$ and $u(x, t) = -1$).

   [*Note: Here it is sufficient to show linear stability of the local equilibria.*]