ADE Exam, Spring 2020

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the SIR model for an epidemic outbreak in a city, which S is the number of susceptible people in the population, I is the infected number and R is the resistant/recovered number. N is the total population size. The standard SIR model is a dynamical system for each variable:

$$\begin{aligned} \frac{dS}{dt} &= -\beta\frac{SI}{N} &(1)\\ \frac{dI}{dt} &= \beta\frac{SI}{N} - \kappa I &(2)\\ \frac{dR}{dt} &= \kappa I. &(3)\end{aligned}$$

Here β and κ are respectively transmission and recovery rates, and we will assume throughout that the starting number of infecteds, $I(0)$ is non-zero.

(a) The constant $R_0 = \beta/\kappa$ is the basic reproduction number for the spread of the disease. Prove that if $R_0 > 1$ then conditions exist in which the number of infected can increase for some period of time.

(b) Prove that for $R_0 < 1$ the number of infected people will decrease over time.

(c) For $R_0 > 1$, What is the critical $S = S_0$ below which the rate of change of infected population is negative?

(d) The city is able to change the R_0 to be less than one with social distancing, thus reducing the rate at which new infections occur. Prove that if the city reverts back to the original social connectivity (original $R_0 > 1$) then after some additional time the total number of people who have been infected at some point in time (including before, during, or after social distancing) will be larger than $N - S_0$.

2. Solve for the Green's function G that satisfies

$$-\frac{\partial}{\partial x}\left(\beta(x)\frac{dG}{dx}(x)\right) = \delta(x-\hat{x}), \quad (x,\hat{x}) \in (0,1)^2 \qquad (4)$$

$$G(0) = 0, \ \frac{dG}{dx}(1) = 0 \qquad (5)$$

with $\beta(x) = \begin{cases} 1, & x < \frac{1}{2} \\ 2, & x > \frac{1}{2} \end{cases}$.

3. Suppose that $u(x,t)$ is a $\mathcal{C}$ and piecewise $\mathcal{C}^{2,2}$ solution of the forced wave equation:

$$u_{tt} = \Delta u + f(x,t) \quad , \quad x \in \mathbb{R}^3 \quad , \quad t > 0$$

with initial conditions $u(x,0) = u_t(x,0) = 0$. Suppose moreover that the forcing function $f(x,t)$ is compactly supported on the spherical shell $1 \leq x \leq 2$.

(a) Find the compact support of u at time $t > 0$. If you make use of any results about the region of influence for wave equation then you should prove them.

(b) Now consider a specific spherically symmetric forcing function, $f \equiv f(r)$ for $0 < t \leq 1$, where $r = ||x||$, and:

$$f(r) = \begin{cases} \dfrac{1}{r} & \text{if } 1 < r < 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find an explicit formula for u for $0 < t \leq 1$.

Hint: You may find it useful to make use of the expression for the Laplacian of a spherically symmetric function:

$$\Delta u(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) .$$

4. $u(x,t)$ is a $\mathcal{C}^{2,1}$ solution of the equation:

$$u_t = \Delta u + u^2 - u \quad , \quad x \in U \; , \quad t > 0 \; ,$$

where U is a bounded open domain, and its boundary ∂U is piecewise smooth. Assume that $u|_{\partial U} = 0$ and $u(x,0) = g(x)$ where $0 \leq g(x) \leq \kappa < 1$. Prove:

(a) $0 \leq u(x,t) \leq \kappa$,

(b) $\int_U u(x,t)^2 \, dx \leq \left(\int_U g(x)^2 \, dx\right) \, e^{2(\kappa-1)t}$.

5. Consider the heat equation

$$u_t = \Delta u, \quad u_{t=0} = u_0$$

with initial data $0 \leq u_0 \leq M$, and $u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. (Recall that the notation $\mathcal{C}_0^\infty$ means that a function is smooth and decays to 0 at ∞).

(a) Prove that for any $T > 0$, the solution $u(x,t)$ satisfies $0 \leq u(x,t) \leq M$. Hint - use the explicit form of the solution of the Heat equation.

(b) Consider a nonnegative function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Define $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(x/\epsilon)$ with $\int_{\mathbb{R}^n} \phi \, dx = 1$, and define $u_\epsilon(x) = \phi_\epsilon * u_0$ with u_0 defined as above. Prove that $u_\epsilon(x)$ satisfies $0 \leq u_\epsilon(x) \leq M$.

6. If U is the n-cube: $\{x \in \mathbb{R}^n : -1 < x_i < 1,\ i = 1, 2, \dots n\}$, show that the $\mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ solutions of the equation:

$$-\Delta u = -x \cdot \nabla u + 1$$

depend continuously on the boundary data. Specifically, show if ∂U is the boundary of the cube, and u_1, u_2 are solutions with $u_i(x)|_{\partial U} = g_i(x)$, then given any $\epsilon > 0$ there exists $\delta > 0$ such that if $||g_1 - g_2||_\infty < \delta$ on ∂U, then $||u_1 - u_2||_\infty < \epsilon$ on $\bar{U}$.

7. Solve Burgers' Equation:

$$u_t + u u_x = 0,\ x \in \mathbb{R}, t > 0$$

with initial conditions:

$$u(x, 0) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}.$$

8. Consider the energy functional

$$e[\phi] = \int_0^1 \frac{1}{2} \left(\log \left(\frac{d\phi}{dx} \right) \right)^2 dx - g\, \phi(1)$$

for functions $\phi : [0, 1] \to \mathbb{R}$ with $\phi(0) = 0$. Assume g is a constant. How many (smooth) global minimizers does the functional have if:

   (a) $g \leq 0$?

   (b) $g > 0$?