Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 1999

MS: Do any 4 of the following 8 problems
Ph.D.: Do any 6 of the following 8 problems.

1. Suppose that $\Delta u = 0$ in the weak sense in R^n and that there is a constant C such that

$$\int_{\{|x-y|<1\}} |u(y)|dy < C, \ \forall x \in R^n.$$

Show that u is constant.

2. Given that $K_a(x-y)$ and $K_b(x-y)$ are the kernels for the operators $(\Delta - aI)^{-1}$ and $(\Delta - bI)^{-1}$ on $L^2(R^n)$, where $0 < a < b$. show that $(\Delta - aI)(\Delta - bI)$ has a fundamental solution of the form $c_1 K_a + c_2 K_b$.

Use the preceding to find a fundamental solution for $\Delta^2 - \Delta$, when $n = 3$.

3. Consider the elliptic (the matrix a^{ij} is positive definite) operator with smooth coefficients,

$$Lu = -\sum \partial_{x_j}(a^{ij}(x)\partial_{x_i} u) + c(x)u,$$

in the bounded domain U with smooth boundary ∂U. Assuming the Neumann condition, $\partial u/\partial \nu = 0$ on ∂U, show that L must have a negative eigenvalue if

(Note here $\partial u/\partial \nu$ means $\sum_{i,j} \nu_i a^{ij} \partial_{x_j} u$)

$$\int_U c(x)dx \leq 0 \text{ and } c(x) \neq 0$$

for some $x \in U$.

4. Consider the Cauchy problem, $u_t + a(x)u_x = 0$, $u(x,0) = f(x)$ for $x \in R$. Give an example of an (unbounded) smooth $a(x)$ for which the solution of the Cauchy problem is not unique.

5. In two spatial dimensions, consider the differential equation

$$\partial_t u = -\varepsilon \Delta u - \Delta^2 u$$

with periodic boundary conditions on the unit square $[0, 2\pi]^2$.

(i) If $\varepsilon = 2$ find a solution whose amplitude increases as t increases.

(ii) Find a value ε_0, so that the solution of this PDE stays bounded as $t \to \infty$, if $\varepsilon < \varepsilon_0$.

6. For the system

$$\partial_t \rho + \partial_x u = 0$$

$$\partial_t u + \partial_x(\rho u) = \partial_x^2 u$$

look for traveling wave solutions of the form $\rho(x,t) = \rho(y = x - st)$, $u(x,t) = u(y = x - st)$. In particular

(i) Find a first order ODE for u.

(ii) Show that this equation has solutions of the form $u(y) = u_0 + u_1 \tanh(\alpha y + y_0)$, for some constants u_0, u_1, α, y_0.

7. Consider the differential operator

$$L = (d/dx)^2 + 2(d/dx).$$

The domain is $x \in [0,1]$, with boundary conditions $u(0) = u(1) = 0$.

(i) Find a function $\phi = \phi(x)$ for which L is self-adjoint in the norm

$$||u||^2 = \int_0^1 u^2 \phi dx$$

.

(ii) If $a < 0$ show that $L + aI$ is invertible.

(iii) Find a value of a, so that $(L + aI)u = 0$ has a nontrivial solution.

8. Let $u = u(x,t)$ solve the following PDE in two spatial dimensions

$$-\Delta u = 1$$

for $r < R(t)$, in which $r = |x|$ is the radial variable, with boundary condition

$$u = 0$$

on $r = R(t)$. In addition assume that $R(t)$ satisfies

$$\frac{dR}{dt} = -\frac{\partial u}{\partial r}(r = R)$$

with initial condition $R(0) = R_0$.

(i) Find the solution $u(x,t)$.

(ii) Find an ODE for the outer radius $R(t)$, and solve for $R(t)$.